BRIEF COURSE IN ANALYTICS

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PREFACE

The saying that you must be able to think in a language before you understand it is peculiarly applicable to mathematics, because in order to understand any course in mathematics one must think in terms of the ideas and principles of that course. It is the hope of the authors that this text on analytics will encourage and help the student to develop his powers of reasoning and will lead him to an appreciation of the beauties of the subject. To this end many problems have been included in the text, some of which are quite easy, others difficult. By solving the former, the student will gain confidence in himself and will be able to apply his knowledge in solving the latter. We feel that his success in analyzing and solving the more difficult problems is a good measure of his mastery of the course; if the development of the theory has helped him appreciably in this direction, then our purpose in writing the book has been accomplished.

The text is designed for classes which meet three hours a week, and the material has been so divided that illustrative examples and a set of exercises come at the end of each assignment. While none of the subject matter is new, the customary arrangement of certain topics has been altered in order to give greater unity to the course.

M. A. H., Jr.
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BRIEF COURSE IN ANALYTICS

CHAPTER I

BASIC DEFINITIONS AND THEOREMS

1. The Nature of Analytic Geometry. Previous to this time the student has received mathematical training in algebra and geometry but has had little opportunity, except in the study of graphs, to see how these two branches of learning are connected. In the present course, **analytic geometry,** which may be defined as the study of geometry by means of the analytical methods of algebra, he will express geometric figures and facts about such figures in algebraic terms and will obtain results from equations rather than from the figures themselves. This beautiful and very powerful method of determining properties of lines and curves by means of equations is rich in historical background, a brief survey of which will give our course a fuller meaning.

While the beginnings of mathematics are lost in antiquity, it is known that the Egyptians used geometry as early as 1700 B.C. for the practical purposes of surveying and finding areas. Centuries later this knowledge came to the Greeks, but they were more interested in geometry as a means of advancing logical reasoning and developed the subject along this line. In fact, Plato, whose contributions to the logic and methods employed are invaluable, decreed that no instruments other than the ruler and compasses might be used, on the ground that the value of geometry as an intellectual exercise would otherwise be destroyed.

Of the many writers of this golden period of Greek mathematics, the two whose work has greatest interest for us are

Euclid, and Apollonius of Perga. Euclid flourished about 325 B.C. and left to posterity one of the great works of all times, the *Elements*. It is a summary and arrangement of all mathematical knowledge of his age, and is of especial interest to us because it contains most of the plane geometry taught in our present-day schools. Apollonius, who was known as the "Great Geometer," is thought to have lived during the period 260–200 B.C. His greatest contribution was to the study of sections cut from a cone by a plane passing through it. He called the resulting curves ellipses, hyperbolas and parabolas, even as we do today. We shall study these curves in the present course, but our method of approach will be quite different from that used by Apollonius.

Although some branches of mathematics advanced during the following centuries, notably algebra and trigonometry, it has been said, in effect, that Apollonius brought the subject of geometry to as high a state of perfection as was possible without a more powerful method of approach. Some eighteen hundred years later this new approach was found in the discovery of analytic, or coordinate, geometry. Many mathematicians contributed to its development, but René Descartes, a French philosopher, was the first to publish an account of the method and is regarded as its founder. This method, which, in our present-day terminology, connects the distances of a point from two intersecting lines by means of an equation, was given to the world in 1637 and was a spur to the awakening interest in the study of mathematics.

2. Rectangular Cartesian Coordinates. The system of coordinates which we shall use is the familiar one of trigonometry. The plane is divided into four quadrants, I, II, III, IV (Fig. 1), by two perpendicular lines intersecting at O. The line $X'OX$ is called the x-axis, $Y'OY$ the y-axis, and the two together the *coordinate axes*. The point O is called the *origin*. Any point in the plane, such as P, is located by measuring a known distance, x, from the y-axis and a known distance, y, from the x-axis.

The distance from the y-axis is called the *x-coordinate* or *abscissa* of the point, that from the x-axis the *y-coordinate* or *ordinate* of the point, and the two distances taken together and enclosed in a parenthesis (x,y), the *coordinates* of the point. The abscissa is always written first. The origin O corresponds to the zero of our real number system, points to the right of the y-axis having positive abscissas, those to the left, negative. Likewise, points above the x-axis have positive ordinates, those below, negative. Thus the points whose coordinates are (2,5), (−3,2), (−4,−5) and (1,−3) are in the first, second, third and fourth quadrants, respectively. Arrowheads are placed on the axes to show positive direction.

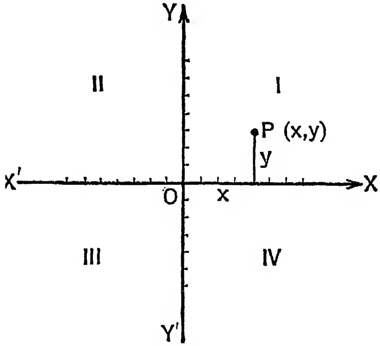


FIG. 1

While the method of locating points in a plane by means of distances and directions from two intersecting lines was known as early as the time of Apollonius, or even before, it is called Cartesian in honor of Descartes who used a modified form of our present system in his work. It is called rectangular since the axes meet at right angles. Obviously a satisfactory system of coordinates could be based on lines intersecting at any angle.

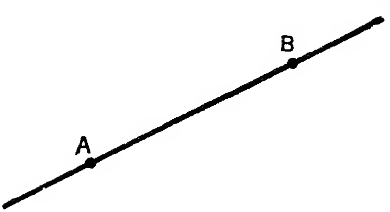


FIG. 2

3. Directed Line Segments. A line segment to which a positive or negative direction has been assigned is called a *directed line segment*. Thus, if AB in Fig. 2 represents the length of the segment from A to B, then BA will represent the length of the

segment measured in the opposite direction. That is,

$$BA = -AB, \qquad \text{or} \qquad AB + BA = 0.$$

This idea has already been used in setting up our coordinate system, because, by definition, an abscissa has positive or negative direction according to whether it is measured to the right or left of the y-axis. Also, an ordinate is positive when measured up from the x-axis, negative when measured down. If we now agree that any line drawn parallel to one of the coordinate axes is to have the same direction as that axis, we can derive a relationship which will be of great importance in what follows.

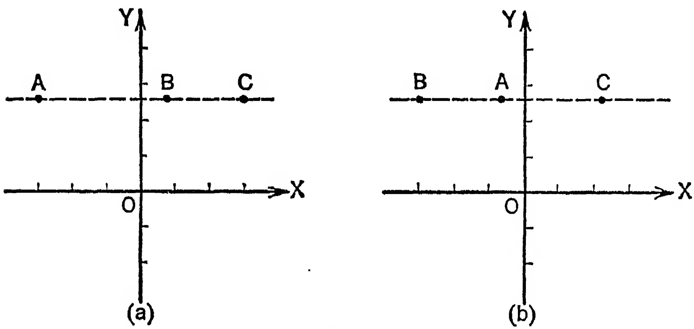


FIG. 3

Figure 3 shows two arrangements of three points A, B, and C on a line parallel to the x-axis. For reasons shown below

$$AB + BC = AC \tag{1}$$

in both cases.

In (a), the segments AB and BC have the same sign and their sum is the positive number AC. In (b), AB and BC are different in sign but BC is the greater, and again their sum is AC. There are four other arrangements of A, B, C, and the student should satisfy himself that the given relation is valid in these cases also.

By revolving the line through 90° the points will lie on a line parallel to the y-axis and equation (1) is again true. Hence we may say that the relation $AB + BC = AC$ is true for all relative positions of A, B, and C on a line parallel to either of the coordinate axes.

EXERCISES

1. Plot the points (2,3), (−5,8), (6,−2), (10,5), (−8,−2), (4,−3) and (−5,2).

2. Plot the points (−4,3), (2,3), (2,−3), (−4,−3) and connect them in the given order by straight lines. What figure is obtained?

3. Connect the points (0,0), (0,2), (8,2), (8,0) in the given order by straight lines. What is the resulting figure?

4. Connect the points (−5,0), (−3,4), (0,5), (3,4), (5,0) in the given order by a smooth curve. What does the figure look like?

5. What is the abscissa of a point on the y-axis? the ordinate of a point on the x-axis?

6. What are the coordinates of the origin?

7. Where do all points lie whose ordinates are 3? whose abscissas are −4?

8. One end of a line segment drawn parallel to the y-axis is (3,2). If the segment is bisected by the x-axis, what are the coordinates of the other end?

9. One end of a line segment drawn parallel to the x-axis is (3,−4). If the segment is trisected by the y-axis, what are the coordinates of the other end?

10. The origin is the middle point of a line joining (−2,3) and a second point. What are the coordinates of the second point?

11. Where are all points whose ordinates and abscissas are equal? whose ordinates are the negatives of the abscissas?

12. Draw the triangle whose vertices are (6,5), (0,−3), (4,−2); whose vertices are (−2,3), (1,−4), (5,0).

13. Draw a line parallel to the x-axis and select four points on it. Beginning at the left, label these B, A, D, C. Show that $AD = AB + BC + CD$.

14. What are the coordinates of the vertices of a square of side 6 if the center of the square is at the origin and its sides are parallel to the coordinate axes?

15. Three vertices of a rectangle are (0,0), (5,0) and (0,3). What are the coordinates of the fourth vertex?

16. Three vertices of a parallelogram are (−5,4), (0,−2) and (1,4). What are the coordinates of the fourth vertex?

17. The center of a square is (4,5) and two vertices are (1,5) and (4,8). What are the coordinates of the other vertices?

4. The Distance between Two Points. In finding the distance between two points P_1 and P_2, whose coordinates are respectively (x_1,y_1) and (x_2,y_2), there are two cases to consider. The first is when the given points are on a line parallel to one of the coordinate axes, and the second, when this position is not held.

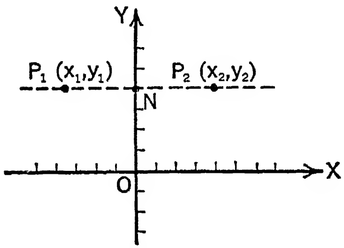


FIG. 4

When P_1 and P_2 are on a line parallel to the x-axis (Fig. 4), we know that $y_1 = y_2$, and relation (1) established in the last article shows us that *the distance from P_1 to P_2 is*

$$P_1P_2 = P_1N + NP_2 = NP_2 - NP_1$$

for all positions of P_1 and P_2. But $NP_2 = x_2$ and $NP_1 = x_1$.

Hence $$P_1P_2 = x_2 - x_1.$$

In like manner, if the points are on a line parallel to the y-axis

$$P_1P_2 = y_2 - y_1.$$

These results may be stated in the following words. *The distance between two points on a line parallel to the x-axis is the abscissa of the terminal point minus the abscissa of the initial point. If the points are on a line parallel to the y-axis, the distance between them is the ordinate of the terminal point minus the ordinate of the initial point.*

Figure 5 illustrates the second, and general, case where the points P_1 and P_2 may be anywhere in the plane. To find the undirected distance, d, between the two points, draw a line through P_1 parallel to the x-axis and a line through P_2 parallel to the y-axis. These lines meet in the point Q whose coordinates are (x_2,y_1). Using the results found above, we have

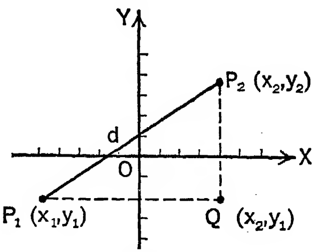


FIG. 5

$$P_1Q = x_2 - x_1 \qquad \text{and} \qquad QP_2 = y_2 -$$

Then by means of the Pythagorean theorem,

$$d^2 = (P_1Q)^2 + (QP_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and the required distance, in terms of the coordinates of the end points, is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (2)$$

Since we are interested only in the numerical measure of the distance, the radical is taken with the positive sign.

While it is advisable to use the results of the first case given above when the points are on a line parallel to an axis, the distance between such points may be found by means of equation (2). If such is done, the sign of the radical is determined by the direction of the axis.

It is to be observed that $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ are always positive, and therefore it is immaterial whether (x_1,y_1) or (x_2,y_2) is taken as the initial point when using (2) to find the distance between points.

EXAMPLE 1. Find the distance from (8, −3) to (2, −3).

Here $x_1 = 8$, $y_1 = -3$; $x_2 = 2$, $y_2 = -3$. Since $y_1 = y_2$, we know that the points are on a line parallel to the x-axis. Hence

$$d = 2 - 8 = -6.$$

Thus the points are six units apart, and the negative sign shows that the distance is measured from right to left.

EXAMPLE 2. Find the distance between the points (3, −8) and (−6, 4).

Using equation (2), we have

$$d = \sqrt{(3+6)^2 + (-8-4)^2} = \sqrt{81 + 144} = \sqrt{225} = 15.$$

5. Division Point of a Line Segment. The coordinates of the point dividing a line segment P_1P_2 in the ratio r_1/r_2 may be found as follows. In Fig. 6 let the *initial point* P_1 and the *terminal point* P_2 have coordinates (x_1,y_1) and (x_2,y_2), respectively. Let P_0, with coordinates (x_0,y_0), be the point on the line joining P_1 and P_2 such that

$$\frac{P_1P_0}{P_0P_2} = \frac{r_1}{r_2}.$$

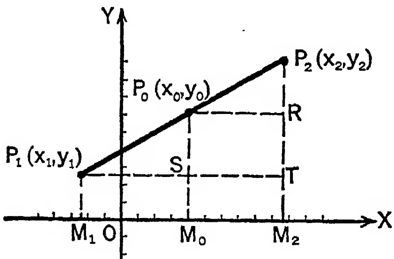


FIG. 6

Draw the ordinates M_1P_1, M_0P_0 and M_2P_2, and through P_1 and P_0 draw lines P_1ST and P_0R parallel to the x-axis. We then have $P_1S = x_0 - x_1$, $P_0R = x_2 - x_0$, $SP_0 = y_0 - y_1$, and $RP_2 = y_2 - y_0$. Since triangles P_1SP_0 and P_0RP_2 are similar, we may write

$$\frac{P_1S}{P_0R} = \frac{P_1P_0}{P_0P_2}, \quad \text{or} \quad \frac{x_0 - x_1}{x_2 - x_0} = \frac{r_1}{r_2}.$$

Solving for x_0, we have $x_0 = \dfrac{x_1r_2 + x_2r_1}{r_1 + r_2}$. (3)

In like manner, $\dfrac{SP_0}{RP_2} = \dfrac{P_1P_0}{P_0P_2}$, or $\dfrac{y_0 - y_1}{y_2 - y_0} = \dfrac{r_1}{r_2}$,

and
$$y_0 = \frac{y_1r_2 + y_2r_1}{r_1 + r_2}. \tag{4}$$

In the case considered above P_0 *lies between* P_1 *and* P_2, that is, P_1P_0 and P_0P_2 have the same sign. If P_0 is not between P_1 and P_2 but falls upon P_1P_2 extended, and hence divides it *externally*, P_1P_0 and P_0P_2 differ in sign and the ratio r_1/r_2 is negative.

When the division point P_0 is the mid-point of the segment P_1P_2 and therefore $r_1 = r_2$, equations (3) and (4) reduce to

$$x_0 = \frac{x_1 + x_2}{2}, \qquad y_0 = \frac{y_1 + y_2}{2}. \tag{5}$$

EXAMPLE 1. Find the coordinates of the point which is two-thirds of the way from $(-3,5)$ to $(6,-4)$.

Let P_1 be $(-3,5)$ and P_2 be $(6,-4)$. Then if P_0 is the desired point, $P_1P_0/P_0P_2 = \frac{2}{1}$ and

$$x_0 = \frac{(-3)(1) + (6)(2)}{2 + 1} = 3,$$

$$y_0 = \frac{(5)(1) + (-4)(2)}{2 + 1} = -1.$$

Hence the coordinates of the point are $(3,-1)$.

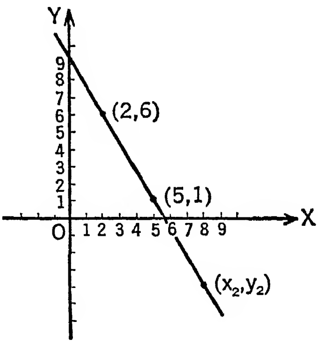


FIG. 7

EXAMPLE 2. In Fig. 7, (5,1) is the mid-point of a segment joining (2,6) and (x_2,y_2). Find (x_2,y_2).

Using (5), with $(x_1,y_1) = (2,6)$ and $(x_0,y_0) = (5,1)$, we have

$$5 = \frac{2 + x_2}{2}, \qquad 1 = \frac{6 + y_2}{2}.$$

Solving for x_2 and y_2, we find $(8,-4)$ to be the coordinates of the required point.

EXERCISES

1. Find the directed distance from $(-5,2)$ to $(3,2)$; from $(-5,2)$ to $(-5,-4)$.

2. If A, B and C have coordinates $(-2,3)$, $(5,3)$ and $(0,3)$, respectively, show that $AB + BC = AC$.

3. Find the coordinates of the mid-point of the segment joining $(-5,8)$ and $(2,-4)$.

4. Draw the triangle whose vertices are $(-4,-3)$, $(8,-3)$ and $(8,5)$. What is its area?

5. Find the mid-points of the sides of the triangle given in Exercise 4.

6. Draw a triangle by joining the mid-points found in Exercise 5 and find its area. Does this area have any relation to the area found in Exercise 4?

7. Find the area of the triangle whose vertices are (a,b), (c,b) and (c,d). Now find the area of the triangle formed by joining the mid-points of the sides of the original triangle. What is the relationship between the two areas?

8. If A, B, C and D have coordinates $(5,-3)$, $(5,2)$, $(5,-1)$ and $(5,8)$, respectively, show that $AB + BC + CD = AD$.

9. A line segment joins the points $(2,8)$ and $(5,-3)$. What is its length? What are the coordinates of its mid-point?

10. Draw the triangle whose vertices are $(-2,1)$, $(1,-3)$, $(4,3)$, and find the lengths of the sides.

11. Connect the following points by straight lines and use the distance formula to show that the resulting figures are as indicated.

a. $(-4,3)$, $(4,-3)$, $(3\sqrt{3},4\sqrt{3})$—an equilateral triangle.
b. $(2,3)$, $(6,8)$, $(7,-1)$—an isosceles right triangle.
c. $(2,1)$, $(3,-2)$, $(6,-1)$, $(5,2)$—a square.
✓d. $(-3,-2)$, $(0,2)$, $(5,2)$, $(2,-2)$—a rhombus.

12. Show that the following points are vertices of right triangles.

a. $(6,3)$, $(4,-3)$, $(2,1)$.
b. $(-2,3)$, $(3,8)$, $(1,2)$.
c. (a,b), $(a + c,b + d)$, $(a - d,b + c)$.

13. Find the lengths of the diagonals of the quadrilateral whose vertices are $(-3,-2)$, $(4,-3)$, $(2,8)$ and $(-4,2)$.

14. Find the coordinates of the points which divide the line segment joining $(-6,-4)$ and $(8,10)$ into three equal parts.

15. A circle passes through the points $(2,1)$, $(-2,5)$ and $(-2,-3)$. Show that $(-2,1)$ is its center.

16. Show that $(3,\frac{1}{2})$ lies on the perpendicular bisector of the line joining $(-2,3)$ and $(4,6)$. Do you know the coordinates of another point which lies on this bisector?

17. Find the coordinates of the point which is three-fourths of the distance from $(6,-2)$ to $(2,6)$.

18. Prove that the ratio r_1/r_2 is negative when P_0 divides P_1P_2 externally.

19. If P_0 divides P_1P_2 externally, where will P_0 be located relative to the segment P_1P_2 when the numerical value of r_1 is greater than the numerical value of r_2? when the numerical value of r_1 is less than the numerical value of r_2?

20. Find the coordinates of the point which divides the segment from $(-3,2)$ to $(5,8)$ externally in the ratio 2 to 3.

21. Find the coordinates of the point which divides the segment from $(-4,-5)$ to $(5,-1)$ externally in the ratio 5 to 3.

22. If the points $(-4,7)$, $(1,6)$ and $(2,-4)$ are the mid-points of the sides of a triangle, what are the coordinates of its vertices?

23. If an equilateral triangle has two vertices $(-4,0)$ and $(4,0)$, what are the coordinates of the third vertex?

24. Show that the line which joins the mid-points of two sides of a triangle with vertices $(2,1)$, $(7,3)$, $(5,-4)$ is one-half the length of the third side.

25. Solve Exercise 24 for the general case; that is, when the vertices are (a,b), (c,d) and (e,f).

26. Find the lengths of the medians of a triangle whose vertices are $(3,4)$, $(6,2)$ and $(10,8)$.

6. Inclination and Slope. The angle, less than 180° and measured counter-clockwise, which a line makes with the positive direction of the x-axis is called the **inclination** of the line. The tangent of this angle is called the **slope** of the line. If we designate the angle by α and the slope by m, then

$$m \quad \tan \alpha.$$

Since line l_1 (Fig. 8) makes an *acute angle*, α_1, with the positive direction of the x-axis, it follows that

$$m_1 = \tan \alpha_1$$

is positive. For this reason l_1 is said to have **positive slope.** In like manner, since α_2 is obtuse, $m_2 = \tan \alpha_2$ is negative and l_2 is said to have **negative slope.** We may conclude, therefore, that *a line which rises from left to right has positive slope and one that descends from left to right has negative slope.*

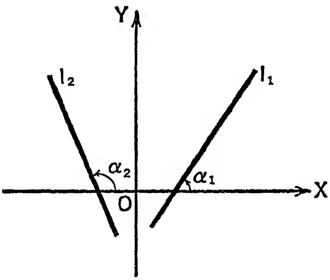


FIG. 8

Since $\tan 0° = 0$ and $\tan 90°$ is undefined, a line parallel to the x-axis has *zero slope* and a line perpendicular to the x-axis is said to have *no slope.* In the latter case, some writers speak of the slope as being infinite. The slope of a line through two points such as P_1 and P_2 (Fig. 9) is readily expressed in terms of the coordinates of these points. That is, for both (a) and (b)

$$m = \tan \alpha = \frac{CP_2}{P_1C} = \frac{y_2 - y_1}{x_2 - x_1}, \tag{6}$$

provided $x_1 \neq x_2$.

It is to be observed that the relation

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

is always true, whether the slope is positive as in (a) or negative as in (b). Hence we may say that *the slope of a line not parallel*

to the y-axis and passing through the points P_1 and P_2 remains the same whether the line is directed from P_1 to P_2 or from P_2 to

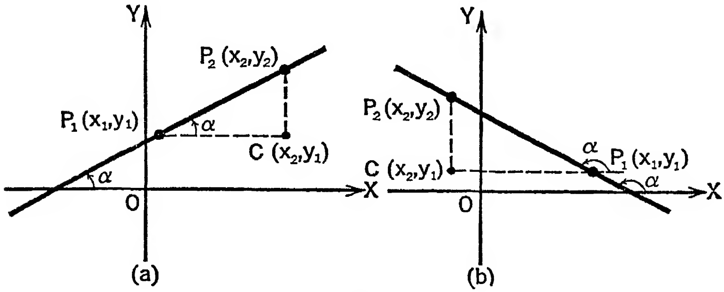


FIG. 9

P_1, and is equal to the difference of the ordinates divided by the corresponding difference of the abscissas.

EXAMPLE. If the points A, B and C, with coordinates $(-2,3)$, $(5,8)$ and $(7,-4)$, respectively, are the vertices of a triangle, find

(a) the slope of the side AB,

(b) the length of the side BC,

(c) the coordinates of the point two-thirds of the distance from B to the mid-point of the opposite side.

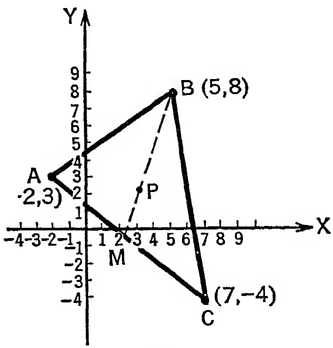


FIG. 10

Using Fig. 10, we have:

(a) The slope of the line AB is given by

$$m = \frac{8-3}{5+2} \quad \frac{5}{7}.$$

While this value was found by taking the direction from A to B, it is equally true that

$$m = \frac{3-8}{-2-5} = \frac{-5}{-7} = \frac{5}{7}$$

where the direction is from B to A.

(b) The length of the side BC is found by means of equation (2), that is,

$$BC = \sqrt{(7-5)^2 + (-4-8)^2} = \sqrt{4+144} = \sqrt{148} = 2\sqrt{37}.$$

(c) By using equations (5) the coordinates of M, the mid-point of the side opposite vertex B, are found to be

$$x_0 = \frac{7-2}{2} = \frac{5}{2}, \qquad y_0 = \frac{-4+3}{2} = -\frac{1}{2}.$$

Now if P is the point two-thirds of the way from B to M, $BP/PM = \frac{2}{1}$ and the coordinates of P are given by

$$x_0 = \frac{(5)(1) + (\frac{5}{2})(2)}{2+1} = \frac{10}{3}, \qquad y_0 = \frac{(8)(1) + (-\frac{1}{2})(2)}{2+1} = \frac{7}{3}.$$

The line BM is a *median* of the triangle, and the point P is called the *center of gravity* of the triangle.

7. Parallel and Perpendicular Lines. *If two lines with slopes m_1 and m_2 are parallel, their slopes are equal; and conversely.* Thus (Fig. 11), if l_1 is parallel to l_2, then $\alpha_1 = \alpha_2$ and $m_1 = \tan\alpha_1$ equals $m_2 = \tan\alpha_2$. Conversely, if $m_1 = m_2$, then $\alpha_1 = \alpha_2$ and the lines are parallel.

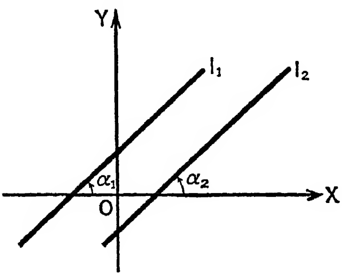


FIG. 11

If two lines with slopes m_1 and m_2 are perpendicular, their slopes are negative reciprocals; and conversely. In Fig. 12, let l_1 and l_2, with slopes $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$, respectively, meet at right angles. Since the exterior angle of a triangle equals the sum of the two opposite interior angles, we may write

$$\alpha_2 = 90° + \alpha_1.$$

Then $\tan\alpha_2 = \tan(90° + \alpha_1) = -\cot\alpha_1 = \dfrac{1}{\tan\alpha_1}$ and therefore

$$m_2 = -\frac{1}{m_1},$$

or $\quad m_1 m_2 = -1. \qquad (7)$

The converse may be proved by reversing the steps, assuming that $\alpha_2 > \alpha_1$.

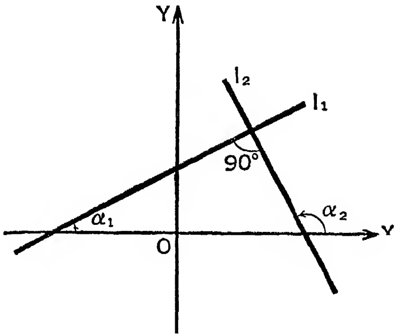


FIG. 12

EXAMPLE. Show that the line joining the points (5,3) and (2,−4) is perpendicular to the line joining the points (−4,2) and (3,−1).

We have

$$m_1 = \frac{3-(-4)}{5-2} \quad 3+4 = \frac{7}{3} \quad \text{and} \quad m_2 = \frac{2-(-1)}{-4-3} = -\frac{3}{7}.$$

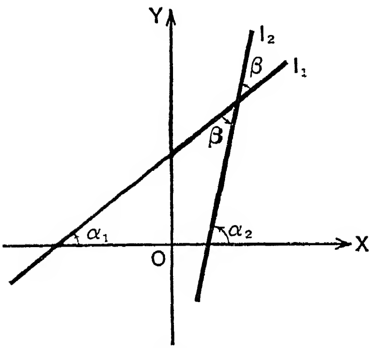


FIG. 13

Therefore, $m_2 = -\dfrac{1}{m_1}$, and the lines are perpendicular.

8. The Angle between Two Lines. The angle between two intersecting lines which do not meet at right angles may be found as follows. In Fig. 13, let l_1 and l_2 be the two lines, and let β be the angle measured counterclockwise from l_1 to l_2. Then

$$\alpha_2 = \alpha_1 + \beta, \qquad \text{or} \qquad \beta = \alpha_2 - \alpha_1,$$

and we may write

$$\tan \beta = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}.$$

But $\tan \alpha_1 = m_1$ and $\tan \alpha_2 = m_2$. Hence the equation may be expressed

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{8}$$

The sign of $\tan \beta$ in equation (8) tells us whether we have found the acute or the obtuse angle between the lines. If $\tan \beta$ is positive, the angle is acute and if negative, it is obtuse. Knowing β, the supplementary angle may be obtained by subtracting it from 180°. In this connection, *it is to be noted that β is measured from l_1 to l_2 and hence m_2 is the slope of the line which is the terminal side of the angle.* Except in specified cases, however, it is immaterial which line is designated as l_2 if we remember that once our choice is made, the angle β remains fixed.

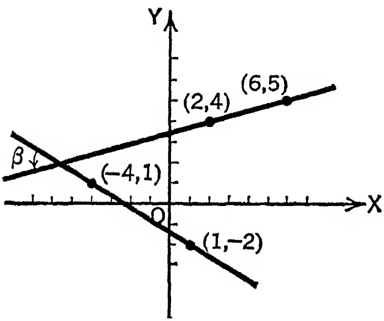


FIG. 14

EXAMPLE. Find the acute angle which the line joining the points $(1,-2)$ and $(-4,1)$ makes with the line joining the points $(2,4)$ and $(6,5)$.

Plotting the points (Fig. 14), we see that m_2 is the slope of the line through the points $(2,4)$ and $(6,5)$. Therefore

$$m_1 = \frac{-2 - 1}{1 + 4} = -\frac{3}{5}, \qquad m_2 = \frac{5 - 4}{6 - 2} = \frac{1}{4},$$

$$\tan \beta = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{3}{20}} = \frac{17}{17} = 1 \qquad \text{and} \qquad \beta = 45°.$$

Now suppose we had not plotted the points and had chosen

the line through $(1,-2)$ and $(-4,1)$ as the terminal side of the angle. Then

$$m_1 = \frac{5-4}{6-2} = \frac{1}{4}, \qquad m_2 = \frac{-2-1}{1+4} = -\frac{3}{5},$$

$$\tan\beta = \frac{-\frac{3}{5}-\frac{1}{4}}{1-\frac{3}{20}} = -\frac{17}{17} = -1 \qquad \text{and} \qquad \beta = 135^\circ.$$

We now obtain the desired angle from the relation $180^\circ - \beta$, that is, $180^\circ - 135^\circ = 45^\circ$.

EXERCISES

1. Find the slope of the line joining the following pairs of points.
   a. $(3,4)$, $(5,9)$. c. $(1,-2)$, $(6,8)$. e. $(-5,-4)$, $(2,-3)$.
   b. $(-3,2)$, $(2,-4)$. d. $(2,5)$, $(3,-6)$. f. $(3.2,1.6)$, $(-5.8,4.6)$.

2. Draw the line which passes through the point $(2,-3)$ with slope $\frac{2}{3}$; with slope $-\frac{1}{2}$.

3. Find the slope and inclination of the line joining (a,b) to (c,b). What can you say of the slope and inclination of a line parallel to the

4. Find the slope of the line that is perpendicular to the line joining $(3,-2)$ and $(-4,-1)$; joining $(-2,8)$ and $(3,6)$.

5. Show that the line through $(1,1)$ and $(-2,3)$ is parallel to the line through $(3,2)$ and $(-3,6)$. Draw a figure.

6. Draw a figure and show that the line through $(3,5)$ and $(-2,3)$ is parallel to the line through $(5,-1)$ and $(-10,-7)$, but is perpendicular to the line through $(2,-1)$ and $(-4,14)$.

7. Prove by means of slopes that the three points $(0,3)$, $(2,6)$ and $(-2,0)$ lie on the same straight line.

8. Find the value of y so that the points $(-1,7)$, $(3,-5)$ and $(-4,y)$ shall lie on the same straight line.

9. Prove that $(2,4)$, $(-2,0)$, $(6,0)$ and $(2,-4)$ are vertices of a square. Draw a figure.

10. Prove that $(-3,4)$, $(9,-5)$ and $(-9,-4)$ are vertices of a right triangle.

11. Draw the quadrilateral whose vertices are $(2,4)$, $(1,5)$, $(-2,2)$, $(-1,1)$, and show that it is a rectangle.

12. A line passes through the point $(-2,3)$ and is perpendicular to the line joining $(-5,-4)$ and $(1,5)$. What is its slope? Draw a figure.

13. The vertices of a triangle are $A(-2,5)$, $B(3,8)$ and $C(6,-4)$. Draw the figure and show that the line joining the mid-points of sides AB and BC is parallel to and one-half the length of side AC.

14. Find the acute angle between the line joining $(-1,3)$ to $(3,5)$ and the line joining $(-2,8)$ to $(-3,5\sqrt{3})$.

15. Find the tangents of the angles of the triangle given in Exercise 13.

16. The angle between two lines is 30° and the slope of one of the lines is $\frac{2}{3}$. What is the slope of the other line?

17. By means of slopes prove that the perpendicular bisector of the line joining $(5,-2)$ to $(7,4)$ passes through $(-3,4)$. Draw a figure.

18. Draw the triangle whose vertices are $(-5,0)$, $(8,2)$, $(4,-3)$ and show that $(\frac{43}{38},\frac{129}{38})$ lies on the perpendicular bisectors of the sides.

19. Show that the triangle whose vertices are $(1,7)$, $(5,-1)$, and $(3+4\sqrt{3}, 3+2\sqrt{3})$ is equiangular by proving that the medians are perpendicular to the sides.

20. What is the angle of intersection of two lines with slopes $-m$ and $(1-m)/(1+m)$?

21. Prove the converse of the theorem stated in Art. 7 regarding perpendicular lines. Also, discuss the case where $\alpha_1 > \alpha_2$.

22. If the relationship $m_1 = -m_2$ holds for two lines, show that these lines and the x-axis form an isosceles triangle.

23. Find the angle of intersection of two lines with slopes $m = \tan \alpha$ and $1/m = \cot \alpha$.

24. Derive the equation which expresses the fact that a line passing through $(3,2)$ is parallel to the line joining $(-4,6)$ and $(5,-3)$.

25. Derive the equation which expresses the fact that a line passing through $(3,2)$ is perpendicular to the line joining $(-4,6)$ and $(5,-3)$.

9. Some Theorems from Elementary Geometry. In proving geometric theorems by the analytical method, the student should bear in mind that the proof must be general; that is,

the figure is not to be restricted beyond the statement of the theorem, and the results are to be expressed in terms of letters. Also, in order to lessen the numerical work the coordinate axes should be used as parts of the figure whenever it is possible to do so without lessening the generality of the proof.

EXAMPLE 1. Prove that the diagonals of a parallelogram bisect each other.

We may, without loss of generality, choose the origin as a vertex and an axis as one side of the parallelogram. Why? We may not, however, use both axes as sides, because our figure would then contain a right angle and the parallelogram would be of special shape.

Choosing the x-axis as a side (Fig. 15), we have O and P_1, with coordinates (0,0) and $(a,0)$, respectively, as two vertices. If P_2, with coordinates (b,c), is taken as the third vertex, then P_3, the fourth vertex, must have coordinates $(a + b,c)$. This is true since the segment P_2P_3 is parallel to and equal in length to the segment OP_1.

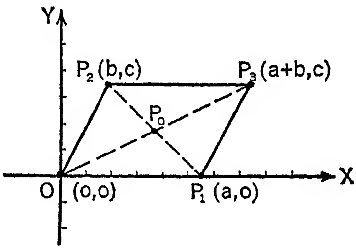


FIG. 15

The theorem will be proved if we can show that P_0 is the mid-point of both diagonals. Using the diagonal OP_3, we find the coordinates of P_0 to be $\left(\frac{a+b}{2},\frac{c}{2}\right)$. Likewise, using the diagonal P_1P_2, the coordinates of P_0 are found to be $\left(\frac{a+b}{2},\frac{c}{2}\right)$. Hence the diagonals bisect each other and the theorem is proved.

EXAMPLE 2. If two medians of a triangle are equal, prove that the triangle is isosceles.

Choose the triangle with a base length of $2a$ units (Fig. 16), having P_1 and P_2, with coordinates $(-a,0)$ and $(a,0)$, respec-

tively, as the base vertices. Let the third vertex, P_3, have coordinates (b,c). Let M_1 and M_2 be the mid-points of sides P_2P_3 and P_1P_3, respectively.

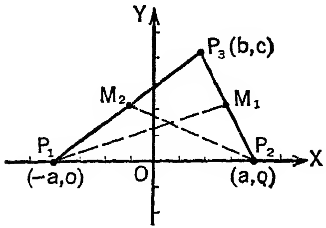


Fig. 16

Consider the medians P_1M_1 and P_2M_2. By hypothesis

$$P_1M_1 = P_2M_2$$

and we must show that

$$P_1P_3 = P_2P_3.$$

This is equivalent to saying that P_3 falls upon the y-axis, that is, that $b = 0$, since P_1 and P_2 are equidistant from O.

The coordinates of M_1 and M_2 are, in order,

$$\left(\frac{a+b}{2},\frac{c}{2}\right) \quad \text{and} \quad \left(\frac{b-a}{2},\frac{c}{2}\right),$$

and therefore

$$P_1M_1 = \sqrt{\left(\frac{a+b}{2}+a\right)^2 + \left(\frac{c}{2}-0\right)^2}$$

$$= \tfrac{1}{2}\sqrt{(b+3a)^2+c^2},$$

$$P_2M_2 = \sqrt{\left(\frac{b-a}{2}-a\right)^2 + \left(\frac{c}{2}-0\right)^2}$$

$$= \tfrac{1}{2}\sqrt{(b-3a)^2+c^2}.$$

Hence

$$\tfrac{1}{2}\sqrt{(b+3a)^2+c^2} = \tfrac{1}{2}\sqrt{(b-3a)^2+c^2},$$

which reduces to $12ab = 0$, or $ab = 0$. Since $a \neq 0$, being one-half the length of the base, b must be equal to zero and therefore P_3 falls upon the y-axis. Thus the specified sides of the triangle are equal and the theorem is proved.

EXERCISES

Prove the following general theorems analytically.

1. The diagonals of a square meet at right angles.

2. The vertices of a right triangle are equidistant from the mid-point of the hypotenuse.

3. The diagonals of a rectangle are equal.

4. A rectangle is a square if the diagonals are perpendicular.

5. The diagonals of an isosceles trapezoid are equal.

6. The line joining the mid-points of two sides of a triangle is parallel to and one-half the length of the third side.

7. The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and one-half their sum.

8. Two medians of an isosceles triangle are equal.

9. The medians and altitudes of an equilateral triangle are the same lines.

10. The lines joining the mid-points of opposite sides of a quadrilateral bisect each other.

11. The lines joining the mid-points of adjacent sides of a square form a square.

12. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

13. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

14. The lines joining the mid-points of adjacent sides of a quadrilateral form a parallelogram.

15. The line joining the mid-point of a side of a parallelogram to an opposite vertex and the diagonal cutting this line trisect each other.

16. The medians of a triangle meet in a point two-thirds the distance from each vertex to the mid-point of the opposite side.

17. The sum of the squares of the distances from two opposite vertices of a rectangle to any point P in the plane of the rectangle is equal to the sum of the squares of the distances from the remaining vertices to the same point.

18. The diagonals of a rhombus meet at right angles.

19. If the sum of the squares of two sides of a triangle equals the square of the third side, the triangle is a right triangle.

20. The perpendicular dropped from the right angle to the hypotenuse of a right triangle is a mean proportional between the segments cut off on the hypotenuse by the foot of the perpendicular.

CHAPTER II

EQUATIONS AND LOCI

10. Introduction. In this chapter we shall consider the two fundamental problems of analytic geometry. These problems are concerned with the concept of the **locus of an equation** and the **equation of a locus**, and may be stated as follows:

1. *Given an equation, to find the corresponding locus and its properties.*

2. *Given a locus defined geometrically, to find the corresponding equation.*

In his study of graphic algebra the student has learned to use plotted points for the purpose of drawing the graph of a simple equation. We shall review this elementary work and extend it in order to become thoroughly familiar with the fundamental concepts of analytic geometry.

11. The Locus of an Equation. By definition, *the locus or graph of an equation in two variables is the curve*[1] *which contains all of the points, and no others, whose coordinates satisfy the given equation.* While this definition is the familiar one of algebra and is quite satisfactory, some writers prefer to think of a curve as the path traced by a moving point and define it in the following terms. *If a variable point*[2] *$P(x,y)$ moves in such a way that its coordinates must always satisfy a given equation, then the curve traced by P is called the locus of the equation; that is, the curve is the locus, or place, of all points, and no others, whose coordinates satisfy the equation.*

[1] In this course we shall consider curves as including straight lines.

[2] It will be observed that the coordinates of a variable point do not have subscripts.

In the following examples we shall give definite illustrations of the concept of the locus of an equation.

Example 1. Suppose the choice of coordinates is such that they must satisfy the equation

$$x = 2.$$

Here the value of y is not restricted and may assume any value whatever. The points of this locus will therefore lie on a straight line 2 units to the right of the y-axis and parallel to it, and no points not on the line will satisfy the equation. The line is known as the locus of the equation, and x 2 is the equation of the line.

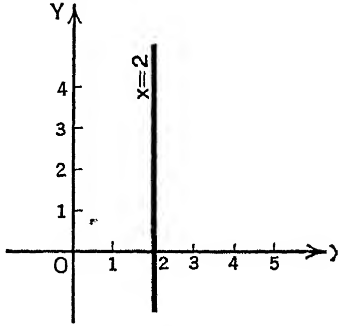


Fig. 17

Example 2. If the values of the coordinates x and y are restricted by the equation

$$x - 2y + 2 = 0,$$

we notice for each arbitrary choice of x, the value of y is definitely determined. Thus, writing the equation in the form $y = \frac{1}{2}x + 1$ and substituting $x = 2$, we find that $y = \frac{1}{2}(2) + 1 = 2$. The other points in the accompanying table are computed in a similar manner. Upon plotting these points, we find that they do not fall at random over the plane but lie on a definite curve, which appears to be a straight line, and this curve is the locus of the equation $x - 2y + 2 = 0$.

x	y
−2	0
0	1
2	2
4	3
6	4

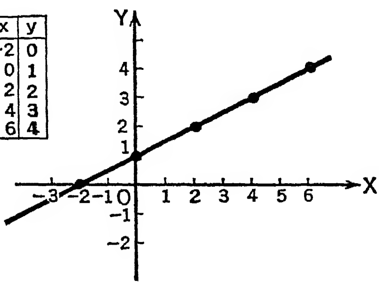


Fig. 18

EXAMPLE 3. Plot the locus of the equation

$$4y^2 - 9x - 18 = 0.$$

x	y
-2	0
-1	± 1.5
0	± 2.12
1	± 2.60
2	± 3
3	± 3.35
7	± 4.5

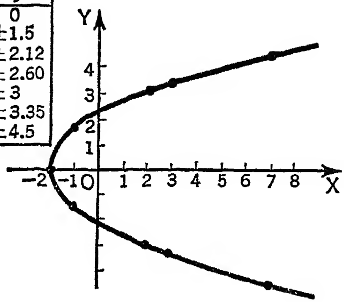


FIG. 19

By solving the equation for y, we obtain

$$y = \pm \tfrac{3}{2}\sqrt{x + 2}.$$

Then, by assigning to x arbitrary values, the corresponding values of y are computed and listed in the table. Plotting these points and connecting them by a smooth curve, we have the locus of the equation.

EXERCISES

Using coordinate paper, draw the locus of the following equations by plotting points.

1. $y = x - 2$.

2. $y = 3x + 2$.

3. $2y = x - 6$.

4. $3x - 2y = 0$.

5. $2x - 3y = 6$.

6. $y - 3 = 0$.

7. $4x + y = 8$.

8. $2y = 3x - 5$.

9. $4x + 3y + 12 = 0$.

10. $y^2 = 8x$.

11. $y^2 + 6x - 18 = 0$.

12. $y^2 + 2x + 4 = 0$.

13. $x^2 = 4y - 12$.

14. $x^2 + 4x + y = 0$.

15. $y^2 - 6y - 4x + 17 \quad 0$.

16. $y = x(x + 2)(x - 1)$.

17. $x^2 - y^2 = 4$.

18. $x^3 - 4 + y = 0$.

19. $9x^2 + 16y^2 = 144$.

20. $x^2 + y^2 = 16$.

21. $xy = 6$.

22. $y = 2x^2 - 8x + 8$.

23. $y(x^2 + 4) = 8$.

24. $y^2(4 - x) = x^3$.

12. Discussion of an Equation. The graph of an equation when drawn by plotting separate points is usually an approximation, since we cannot possibly plot all the points, and the position of a point cannot be accurately located. By a study, or a discussion, of a particular equation, however, geometric prop-

erties of the curve may be determined and this additional information will serve as a check upon the graph. These different properties are considered in the following paragraphs.

1. **Intercepts.** The intercepts of a curve are the directed distances from the origin to the points where the curve cuts the coordinate axes. To find the x-intercept, substitute $y = 0$ in the equation of the curve and solve algebraically for x. To find the y-intercept, substitute $x = 0$ in the equation and solve for y. In order for a curve to cut an axis, the intercept on that axis must be real.

2. **Symmetry.** Two points are *symmetrical with respect to a line*, called the *axis of symmetry*, if the line is the perpendicular bisector of the segment joining the two points. Two points are *symmetrical with respect to a third point*, called the *center of symmetry*, if this third point is the mid-point of the segment joining the two given points.

A curve is said to be symmetrical with respect to a line as an axis of symmetry, or with respect to a point as a center of symmetry, *if each point on the curve has a symmetrical point with respect to the axis*, or center, *which is also on the curve*. Thus, in order for a curve to be symmetrical about the x-axis, to each point of the curve in the first, or in the second, quadrant there must be a symmetrical point in the fourth, or in the third, quadrant which is also on the curve. This discussion leads to the following tests:

a. If an equation remains unchanged when y is replaced by $-y$, the locus is symmetrical with respect to the x-axis;

b. If an equation remains unchanged when x is replaced by $-x$, the locus is symmetrical with respect to the y-axis;

c. If an equation remains unchanged when x is replaced by $-x$ and y by $-y$ at the same time, the curve is symmetrical with respect to the origin.

Thus, $x + y^2 = 5$, $x^2 + y = 5$ and $x^3 + y = 0$ are symmetrical with respect to the x-axis, the y-axis and the origin, respectively.

3. Extent. When we consider an equation in two variables, it is natural to ask whether there are values of one of the variables which will cause the other to become imaginary. Such values might be called *excluded values,* since they do not give real points on the curve. To find these values, we begin by solving the equation for y in terms of x, and for x in terms of y. If either solution gives rise to radicals of even order, the values of the variable which make the expression under the radical negative must be excluded, since the corresponding values of the other variable are imaginary.

EXAMPLE 1. Examine the curve $y^2 = 4x + 4$ for intercepts, symmetry and extent, and then draw the curve.

(a) For $x = 0$, we have $y = \pm 2$ as the y-intercepts. If $y = 0$, we have $x = -1$ as the x-intercept.

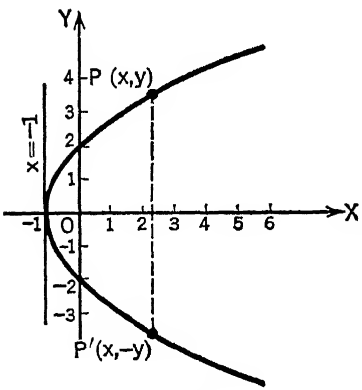


FIG. 20

(b) If we replace x by $-x$, we have $y^2 = -4x + 4$, which is not the same equation as the original, and therefore the curve is not symmetrical about the y-axis. Replacing y by $-y$ gives $(-y)^2 = y^2 = 4x + 4$, or leaves the equation unchanged, and therefore the curve is symmetrical about the x-axis. If both x and y are replaced by $-x$ and $-y$, respectively, the equation is not the same, and hence the curve is not symmetrical about the origin.

(c) If the equation is solved for y, we have $y = \pm 2\sqrt{x + 1}$, which shows that the expression under the radical is positive or zero for $x \geq -1$. This means that y is real for any value of $x \geq -1$, or that the curve lies entirely to the right of the line $x = -1$.

Plotting the intercepts and computing a few additional points, we obtain the curve of Fig. 20.

Example 2. Examine the curve $9x^2 + 25y^2 = 225$ for intercepts, symmetry and extent, and then draw the curve.

(a) When $y = 0$, we have $x = \pm 5$ as the x-intercepts; when $x = 0$, we have $y = \pm 3$ as the y-intercepts.

(b) The equation remains unchanged when x is replaced by $-x$, when y is replaced by $-y$ and when both x and y are

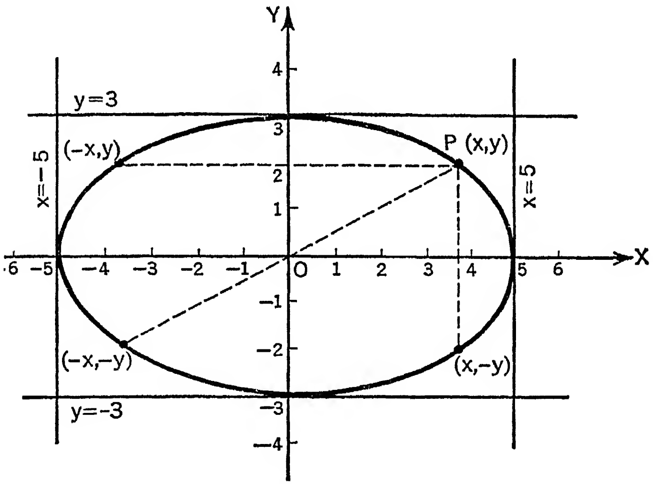


Fig. 21

replaced by $-x$ and $-y$, respectively. This means that the curve is symmetrical about both axes and about the origin.

(c) Solving the equation for y, we have $y = \pm \frac{3}{5}\sqrt{25 - x^2}$. This shows that in order for y to be real, x must not be greater than 5 or less than -5. Similarly, $x = \pm\frac{5}{3}\sqrt{9 - y^2}$ shows that only values of y from -3 to 3, inclusive, will give real values to x. These facts indicate that the curve is closed and lies wholly within the rectangle bounded by the lines $x = \pm 5$ and $y = \pm 3$.

EXERCISES

Discuss the following equations and draw the curves.

1. $y^2 = 4x$.
2. $y^2 = 16x$.
3. $y = 4 - x^2$.
4. $x^2 - 4y + 3 = 0$.
5. $y = x^2 - 4x$.
6. $y = x^2 - 4x + 3$.
7. $y = x^2 - 8x + 16$.
8. $x^2 + y^2 = 16$.
9. $4x^2 + y^2 = 16$.
10. $x^2 - y^2 + 9 = 0$.
11. $x^2 + y - 9 = 0$.
12. $x^2 - x - y = 0$.
13. $4x^2 + 4y^2 = 1$.
14. $9x^2 + 4y^2 = 36$.
15. $9x^2 - 4y^2 = 36$.
16. $x^2 + y^2 + 8x = 0$.
17. $x^2 + y^2 - 8y = 0$.
18. $xy - 6 = 0$.
19. $xy = 8$.
20. $x^2 - y^2 + 6x = 0$.
21. $4x^2 + y^2 + 8x = 0$.
22. $x^2 + 4y^2 + 8y = 0$.
23. $4x^2 + 64x - y^2 = 0$.
24. $4x^2 + y^2 - 16x = 0$.
25. $y = x^3 - 4x$.
26. $y = x^4 - 9x^2$.
27. $3x^2 + 4y^2 = 0$.
28. $8y^2 = 27x^3$.
29. $x^2 - 4xy + 4y^2 = 0$.
30. $y = x(x - 2)(x + 1)$.

13. Infinite Extent of a Curve. It frequently happens that one of the variables of an equation becomes infinite for a finite value of the other variable. In such cases the tracing point of the curve recedes to infinity and in general we have two or more branches of the curve. Since it is important to know such values of the variables in discussing and graphing an equation, we shall give a method for finding them when they exist.

Example. Draw the graph of

$$xy + x - 3y - 4 = 0.$$

Solving the equation for y in terms of x and for x in terms of y, we obtain

$$(1)\ y = \frac{4 - x}{x - 3} \qquad \text{and} \qquad (2)\ x = \frac{3y + 4}{y + 1}.$$

In (1), we observe that as x approaches 3, y becomes infinite and therefore the tracing point of the curve recedes to infinity for this value of x. Likewise, in (2), as y approaches -1, x becomes infinite and the curve recedes to infinity for this value of y. By drawing the lines $x = 3$ and $y = -1$ first (Fig. 22) and then computing a table of values, the curve may be readily drawn.

x	y
0	$-\frac{4}{3}$
1	$-\frac{3}{2}$
2	-2
$2\frac{1}{2}$	-3
3	∞
$3\frac{1}{2}$	1
4	0
5	$-\frac{1}{2}$
6	$-\frac{2}{3}$

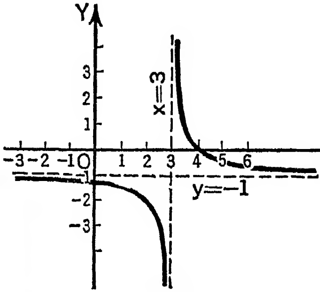


Fig. 22

To summarize the method employed here, we may state the following rule. *Solve the equation for x, and, if the result is a fraction, place the denominator equal to zero and solve for y; solve the equation for y, and, if the result is a fraction, place the denominator equal to zero and solve for x.* In general, the values found by equating the denominators to zero will represent lines along which the curve recedes to infinity.

14. Intersections of Curves. In order that a point shall lie on a curve its coordinates must satisfy the equation of the curve, and conversely. Hence, if two curves intersect, the coordinates of their common points must satisfy both equations. To find the coordinates of such common points, we solve the equations of the two curves simultaneously. If the solutions are real, the curves intersect in real points; if the values of the coordinates are found to be imaginary, the curves do not intersect in real points.

Example. Plot the curves $x^2 + y^2 = 25$ and $3y^2 = 16x$, and find their points of intersection.

Solving the two equations simultaneously, we find that the solutions are $x = 3$, $y = \pm 4$, and $x = -\frac{25}{3}$, $y = \pm \frac{20i}{3}$. This

means that we have found two real points of intersection, (3,4) and (3,−4). The two curves are plotted on the same axes in Fig. 23, showing both points of intersection.

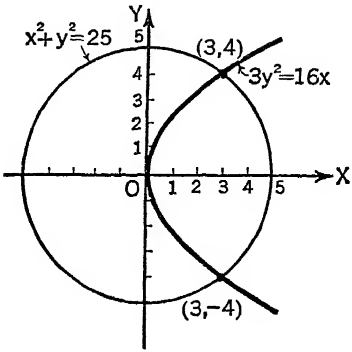


FIG. 23

EXERCISES

Find the lines of infinite extent and plot the graphs of the following equations.

1. $xy - 2y = 8.$
2. $4x + xy = 8.$
3. $xy + 3x = 6.$
4. $xy - 3y = 4 - x.$
5. $4y - xy = 16.$
6. $xy + x - 3y = 0.$
7. $(x^2 - 2x)y = 4.$
8. $(x^2 - 4)(y - 1) = 4.$
9. $xy^2 - 4x = 6y^2.$
10. $(x^2 - 4x)y = 8.$
11. $3xy - 4y = 2x + 3.$
12. $y^2 + 2xy = 8.$
13. $(x - 2)(x - 1)y = 2x + 3.$
14. $xy^2 - 4x = 8.$

Find the points of intersection of the following curves and plot.

15. $x + y = 5,$ $xy = 6.$
16. $y^2 = 2x,$ $2y = x + 2.$
17. $y^2 = 8x,$ $y = 2x - 3.$
18. $x^2 + y^2 = 25,$ $x + 3y = 15.$
19. $x^2 - y^2 = 9,$ $x^2 + y^2 = 25.$
20. $x^2 + 4y^2 = 16,$ $x^2 - y^2 = 9.$

21. $x^2 + y^2 = 9,$

22. $x^2 - 4y^2 = 9,$
$xy = 10.$

23. $x^2 + y^2 = 16,$
$9x^2 + 25y^2 = 225.$

24. $xy = 2,$
$y^2 - x^2 = 3.$

25. $2x^2 + y^2 = 7,$
$x^2 - 2y^2 = -4.$

26. $x^2 + 5y^2 = 20,$
$x^2 + y^2 = 4.$

27. $x^2 + xy = 40,$
$2x - 3y = 1.$

28. $2xy - 3x^2 = -5,$
$x - y = 2.$

15. The Equation of a Locus. The second fundamental problem of analytics is concerned with finding the equation of a locus, or curve, which is defined by means of a geometric property common to all points on the locus, and to no other points. That is, we are given the condition under which a point $P(x,y)$ moves in tracing a locus, and are asked to find an equation in terms of the variables x and y which is satisfied by the coordinates of all points on the locus and by those of no other points. Such an equation is called the *equation of the locus.*

While there are no specific rules for finding such an equation, the following steps should prove useful.

1. If the coordinate axes are not determined by the statement of a given problem, choose them in such a way that the resulting equation will have a simple form. This choice of axes is permissible since the locus is independent of the axes to which it is referred.

2. After constructing the axes, place the point $P(x,y)$, whose locus we wish to determine, in a representative position.

3. Express the condition which P must satisfy in terms of x,y and any constants involved in the definition of the locus. The equation thus obtained, or its simplified form, is the equation of the locus if it contains no variables except x and y and is satisfied by the coordinates of all points on the locus, and by those of no other points.

4. Properties of the locus may be obtained by studying the equation thus obtained.

EXAMPLE 1. Find the locus of a point which is always equidistant from the extremities of the line segment joining the points $(-1,4)$ and $(2,2)$.

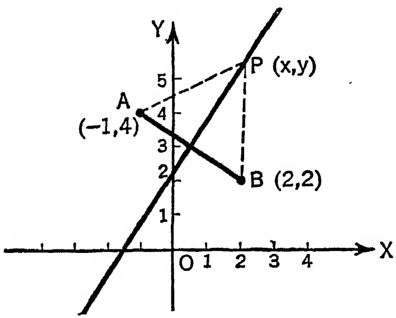


FIG. 24

Here the coordinate axes are given. If in Fig. 24 we let the tracing point be $P(x,y)$, the geometric condition states that

$$PA = PB.$$

Expressing this condition in terms of coordinates, we have

$$\sqrt{(x+1)^2 + (y-4)^2} = \sqrt{(x-2)^2 + (y-2)^2},$$

which becomes by simplifying, $6x - 4y + 9 = 0$. Plotting this equation, we find the locus to be a straight line, the perpendicular bisector of the given segment.

EXAMPLE 2. A point moves so that the sum of its distances from the points $(4,0)$ and $(-4,0)$ is 10 units. Find the equation of the locus.

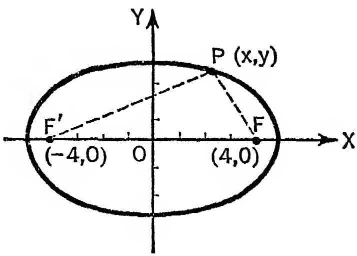


FIG. 25

Let $P(x,y)$ be the tracing point and let F and F' represent the given points (Fig. 25). Then the geometric condition on the point P is that

$$PF' + PF = 10.$$

Now

$$PF' = \sqrt{(x+4)^2 + y^2} \quad \text{and} \quad PF = \sqrt{(x-4)^2 + y^2}.$$

Therefore

$$\sqrt{(x+4)^2+y^2}+\sqrt{(x-4)^2+y^2}=10.$$

Transposing the second radical and squaring both sides, we get

$$x^2+8x+16+y^2=100-20\sqrt{(x-4)^2+y^2}+x^2-8x+16+y^2,$$

which reduces to

$$-25=-5\sqrt{(x-4)^2+y^2}.$$

Squaring again, and reducing, we have $9x^2+25y^2=225$. By drawing the graph of this equation, we find that it is the symmetrical curve, or ellipse, shown in the figure.

EXERCISES

1. A point moves so that it is always 4 units distant from the point $(-2,3)$. Find the equation of its locus.

2. Find the equation of the locus of the point which is always 6 units distant from the origin.

3. Find the equation of the locus of a point which moves so that it is always equidistant from the line $x=-2$ and the point $(2,0)$.

4. A line is 5 units long; one end is at the point $(4,-3)$. Find the locus of the other end.

5. A triangle has a base of length 4 units. The difference of the squares of the lengths of the other two sides is 8 units. Find the equation of the locus of its vertex if the base coincides with the x-axis and is bisected by the origin.

6. A point moves so that the square of its distance from the origin is equal to the square of its distance from the line $x=5$. Find the equation of its locus.

7. If a point moves so that its distance from $(2,0)$ is twice its distance from $(-2,0)$, what is the equation of its locus?

8. Find the equation of the path traced by a point which moves so that it is always 5 units distant from the point $(3,0)$.

9. If one end of a line segment is at the point $(-4,3)$ and if it is 5 units long, find the locus of the other end.

10. A point moves so as to be equidistant from the points $(-2,4)$ and $(4,-2)$. Find the equation of its path.

11. A point moves so as to be always equidistant from the y-axis and the point $(4,0)$. Find the equation of its path and trace the locus from its equation.

12. Find the equation of the curve traced by a point which moves so as to be equidistant from the line $x = 8$ and the point $(-8,0)$.

16. Translation of Axes. In finding the equation of a curve, the coordinates of the tracing point are referred to a set of coordinate axes. If these axes are moved, not only will the coordinates of any fixed point change, but the equation of any fixed curve will likewise change. Sometimes it is desirable to change the axes to which a curve is referred in order to simplify the equation of the curve. When such a change is made and the new axes are drawn parallel to the old, the transformation on the coordinates is known as a **translation.** To obtain the relations which exist between the coordinates of a point referred to one set of axes and the coordinates of the same point referred to a second set of axes which are parallel to the original set, we proceed as follows.

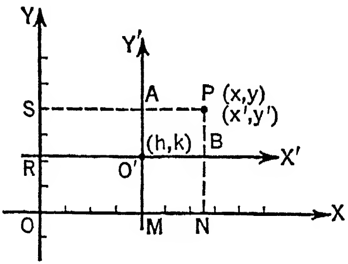


FIG. 26

Let OX and OY be a set of coordinate axes and $O'X'$ and $O'Y'$ be a second set parallel to the first. Then each point in the plane will have two sets of coordinates: (x,y) referred to the original axes and (x',y') referred to the new axes. Let (h,k) be the coordinates of the new origin referred to the old axes, and let P be any point of the plane. Then, from Fig. 26, $x = SP$, $x' = AP$, $h = SA$, $y = NP$, $y' = BP$, and $k = NB$. But

$SP = SA + AP$ and $NP = NB + BP$, and, therefore,

$$x = x' + h \qquad \text{and} \qquad y = y' + k. \tag{9}$$

These formulas are known as *translation formulas* and are true for any position of the point P, or of the axes, so long as the two sets of axes are parallel.

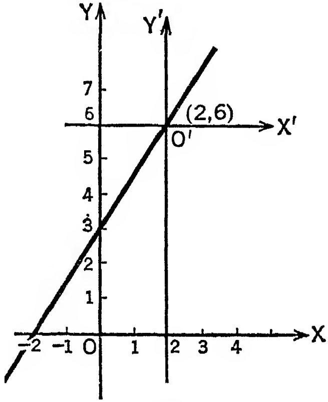


FIG. 27

EXAMPLE 1. Transform the equation $3x - 2y + 6 = 0$ by translating the origin to the point (2,6).

In this case, the formulas of translation become

$$x = x' + 2 \qquad \text{and} \qquad y = y' + 6.$$

Substituting these values in the equation of the given line, we obtain

$$3(x' + 2) - 2(y' + 6) + 6 = 0,$$

or

$$3x' - 2y' = 0$$

as the equation of the line referred to the $O'X'$ and $O'Y'$ axes. This transformation leaves the line unaltered, but, by moving the frame of reference, changes the equation of the line.

EXAMPLE 2. Transform the equation

$$x^2 + y^2 + 6x + 4y - 3 \quad 0$$

by translating the axes to a new origin $(-3,-2)$.

The translation formulas become

$$x = x' - 3 \qquad \text{and} \qquad y = y' - 2.$$

Substituting these values in the equation, we obtain

$$(x' - 3)^2 + (y' - 2)^2 + 6(x' - 3) + 4(y' - 2) - 3 = 0,$$

or

$$x'^2 + y'^2 = 16.$$

The transformation changes the form of the equation but not of the locus. The equation represents a circle of radius 4 and center (0,0) referred to the new axes, or a circle of radius 4 and center $(-3,-2)$ referred to the original axes (Fig. 28).

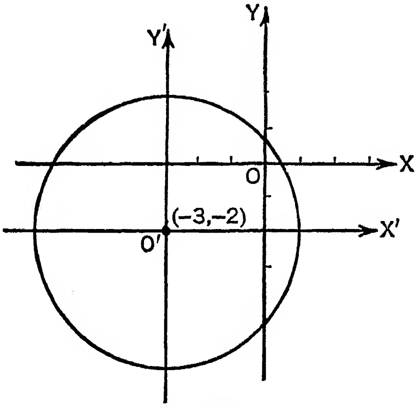


Fig. 28

17. Simplification of Equations by Translation of Axes. One very important use of the translation formulas is to simplify the equation of a given curve by a suitable choice of axes. Two methods of simplification will be illustrated in the following example.

Example. Simplify the equation

$$x^2 + y^2 - 10x + 4y - 7 = 0$$

by removing the first degree terms.

First method. Substitute $x = x' + h$ and $y = y' + k$, and collect terms. This gives

$$x'^2 + y'^2 + (2h - 10)x' + (2k + 4)y' \\ + (h^2 + k^2 - 10h + 4k - 7) = 0.$$

To remove the x' and y' terms, it is necessary that the coeffi-

cients of these terms become zero, that is,

$$2h - 10 = 0 \quad \text{and} \quad 2k + 4 = 0;$$

or

$$h = 5 \quad \text{and} \quad k = -2.$$

Substituting these values in the equation, we obtain

$$x'^2 + y'^2 = 36$$

as the equation of the locus referred to the new axes so chosen as to remove the first degree terms. The new origin is the point $(5,-2)$.

Second method. Completing the squares on the x terms and on the y terms in the equation $x^2 + y^2 - 10x + 4y - 7 = 0$, we may write

$$(x - 5)^2 + (y + 2)^2 = 36.$$

If we let $x - 5 = x'$ and $y + 2 = y'$, the equation becomes

$$x'^2 + y'^2 = 36,$$

where again, the coordinates of the new origin (h,k) are $(5, -2)$.

The two methods of determining the new origin give the same results, but the second method is to be preferred in this case. It should be mentioned that this method is not used when an equation contains an xy-term.

EXERCISES

1. Find the new coordinates of the points $(2,4)$, $(-2,2)$, and $(-2,0)$ if the axes are translated to a new origin at (a) $(4,4)$; (b) $(2,2)$; (c) $(0,-4)$; (d) $(-2,-3)$; (e) $(\frac{5}{2},\frac{3}{2})$.

2. Find the new coordinates of the points $(3,-4)$, $(-2,-3)$, and $(-1,3)$ when the axes are translated to a new origin at (a) $(3,0)$; (b) $(3,4)$; (c) $(\sqrt{2},\sqrt{3})$; (d) $(-1,-2)$; (e) $(2,1)$.

Find the equation of each of the following curves if the axes are translated to the new origin as indicated. Draw the curve and both sets of axes in each case.

3. $2x - 3y = 6$; $(4,1)$. **4.** $5x + y - 10 = 0$; $(2,0)$.

5. $x^2 + y^2 - 6x + 4y - 12 = 0$; $(3,-2)$.

6. $y^2 - 2y - 4x + 5 = 0$; $(\frac{5}{4},1)$.

7. $3y^2 - 12y - 7x - 2 = 0$; $(-2,2)$.

8. $9x^2 + 4y^2 = 36$; $(\frac{1}{3},-2)$. **9.** $y^2 = 4x$; $(1,0)$.

10. $9x^2 + 4y^2 - 54x + 32y + 1 = 0$; $(3,-4)$.

Remove the first degree terms from the following equations by translating the axes. Use the first method.

11. $4x^2 + 4y^2 + 12x - 4y - 6 = 0$.

12. $8x^2 + 8y^2 - 5x + 10y - 2 = 0$.

13. $x^2 + y^2 + 10x - y + 3 = 0$. **14.** $xy - 4x + 3y = 0$.

15. $3x^2 - 3xy - y^2 + 15x + 10y - 24 = 0$.

16. $5x^2 + 2xy + 5y^2 + 10x - 3y + 12 = 0$.

Simplify the following equations by removing the first degree terms by translation. Use the second method and draw the curve, showing both sets of axes.

17. $x^2 + y^2 + 5x + 3y - 4 = 0$. **18.** $x^2 + 8x + 8y^2 = 0$.

19. $y^2 - 8x^2 - 8y + 40 = 0$. **20.** $5x^2 + 3y - 4x = 2 - 5y^2$.

21. $(x - 2)^2 + y^2 - 3y - 5 = 0$. **22.** $(x + 3)^2 + (y - 5)^2 = 4y$.

23. $9x^2 - y^2 + 2y - 10 = 0$. **24.** $5x^2 + 5y^2 + y - 8 = 0$.

25. $5x^2 + 5y^2 + x - 3 = 0$. **26.** $2x^2 + y^2 + 5x = x^2 + y - 4$.

27. $9y^2 - x^2 - 4x - 18y + 30 = 0$.

28. $16x^2 - 9y^2 - 64x - 72y - 224 = 0$.

29. Translate the origin to the point of intersection of the two lines $4x + 3y + 1 = 0$ and $x + y + 1 = 0$, and determine the equations of the lines referred to the new axes.

30. If (h,k) is a point of the line $Ax + By + C = 0$ and the origin is translated to this point, what is the new equation?

CHAPTER III

THE STRAIGHT LINE

18. The Equation of a Line. As a definition we may state that the equation of a straight line is an equation in x and y which is satisfied by the coordinates of every point on the line, and is not satisfied by the coordinates of any point not on the line. The form assumed by the equation will depend upon the data used in determining the line. Thus, if two points are used to determine the line, we shall find that the equation assumes one form; and if one point and a direction are used, we shall obtain a different form. We shall learn that the essential fact about a straight line is that it is determined by two independent conditions, that its equation is of the first degree in the coordinates x and y and may be expressed in several standard forms when desired.

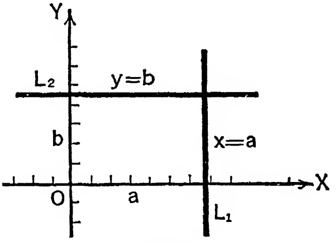


Fig. 29

19. Horizontal and Vertical Lines. When a line is parallel to either axis, its equation may be determined directly from a figure. Thus (Fig. 29), if the line L_1 is drawn parallel to the y-axis and a units from it, then $x = a$ for every point on L_1. Since the equation $x = a$ is satisfied by the coordinates of every point on the line and by those of no other point, it is the equation of the line. The line lies to the right or left of the y-axis according as a is positive or negative.

In like manner, $y = b$ is the equation of L_2, a line parallel to the x-axis.

20. The Point-Slope Form of the Equation of a Line. Let us find the equation of a line L which passes through a fixed point $P_1(x_1,y_1)$ with a given slope m. Take $P(x,y)$ as any other point on the line. Since (x_1,y_1) and (x,y) are on the same line, we may write its slope

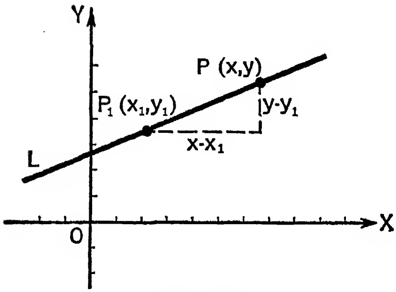


FIG. 30

$$m = \frac{y - y_1}{x - x_1}.$$

Then clearing the fraction, we obtain

$$y - y_1 = m(x - x_1). \quad (10)$$

This equation is true for any position of the point P on the line. Hence we may consider P as a tracing point, for, as it moves, its coordinates will vary but will always satisfy the equation. This first degree equation is called the *point-slope* form of the equation of a line and should be used to write the equation of any straight line which passes through a fixed point with a given slope.

If the coordinates of the given point P_1 are (0,0), equation (10) becomes

$$y = mx$$

and represents a line *through the origin with slope m.*

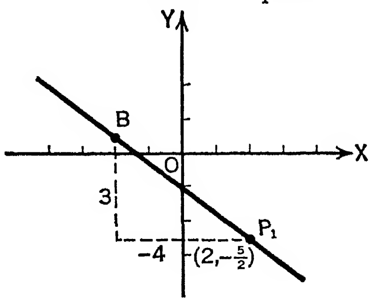


FIG. 31

EXAMPLE. Find the equation of the line which passes through the point $(2,-\frac{5}{2})$ with the slope $-\frac{3}{4}$.

To draw the figure, we plot the given point $P_1(2,-\frac{5}{2})$ and then obtain a second point B (Fig. 31) by measuring from P_1 four units to the left and three units up. The equation of the

line through B and P_1 is $y + \frac{5}{2} = -\frac{3}{4}(x - 2)$ which reduces to

$$3x + 4y + 4 = 0.$$

This equation should be checked by plotting its graph from a table of values to show that the line actually satisfies the given conditions.

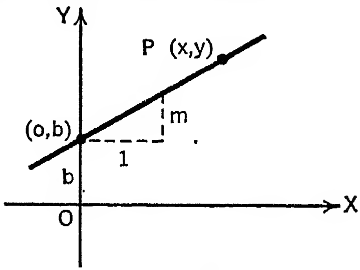


FIG. 32

21. The Slope-Intercept Form of the Equation of a Line. If the y-intercept of a line is b, the coordinates of the point of intersection of the line and the y-axis are $(0,b)$, Fig. 32. To express the equation of a line in terms of its y-intercept b and slope m, we write the equation of the line through the point $(0,b)$ with the slope m, using (10). This gives

$$y - b = m(x - 0),$$

which reduces to

$$y = mx + b. \tag{11}$$

This is called the *slope-intercept* form of the equation of a line. It is of especial importance to notice the form of this equation, because it not only allows us to write down the equation of a line when the y-intercept and slope are known, but it also enables us to find the slope and the y-intercept when the equation is given.

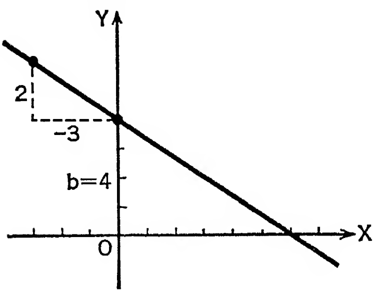


FIG. 33

EXAMPLE. Find the slope and y-intercept of the line whose equation is

$$2x + 3y - 12 = 0.$$

We first solve the equation for y which changes it to the

slope-intercept form, $y = -\frac{2}{3}x + 4$. By comparing this equation with $y = mx + b$ we find that the slope is $m = -\frac{2}{3}$ and the y-intercept is $b = 4$. Using these two quantities the line is easily drawn by measuring 4 units on the positive y-axis and then constructing an angle whose tangent is $-\frac{2}{3}$ (Fig. 33).

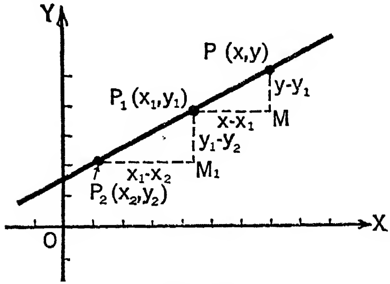


FIG. 34

22. The Two-Point Form of the Equation of a Line. To find the equation of a line determined by two points, we use the method of the previous article. First, we find the slope of the line through the two points; then, by substituting this slope and one of the given points in the point-slope form, we obtain the required equation. Thus, if $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are the given points (Fig. 34), the slope of the line is

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

and by using this slope and one of the points, say (x_1,y_1), in (10), we have the equation

$$y - y_1 = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)(x - x_1).$$

This equation may be written

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \tag{12}$$

and is called the *two-point form* of the equation of a straight line.

Figure 34 indicates that the formula may be derived by using similar triangles. Taking $P(x,y)$ as any point on the line, we

may write

$$\frac{MP}{P_1M} = \frac{M_1P_1}{P_2M_1}, \qquad \text{or} \qquad \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

EXAMPLE. Find the equation of the line determined by the points $(-2,-2)$ and $(5,2)$.

By finding the slope first, we may use the point-slope equation. We have,

$$m = \frac{-2 - 2}{-2 - 5} = \frac{-4}{-7} = \frac{4}{7},$$

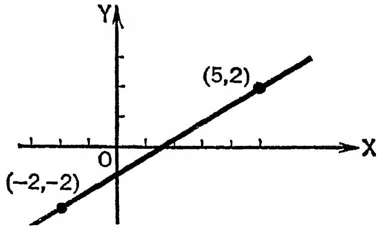


FIG. 35

and, therefore, by using this slope with one of the points, say $(-2,-2)$, we get the equation

$$y + 2 = \tfrac{4}{7}(x + 2), \qquad \text{or} \qquad 4x - 7y - 6 = 0.$$

This same result may be written directly by using the two-point form

$$\frac{y + 2}{x + 2} = \frac{-2 - 2}{-2 - 5} = \frac{-4}{-7}, \qquad \text{or} \qquad \frac{y + 2}{x + 2} = \frac{4}{7},$$

and finally

$$4x - 7y - 6 = 0.$$

The accuracy of the algebraic work should be tested by substituting the coordinates of the given points in the final equation of the line.

23. The Intercept Form of the Equation of a Line. If the x and y intercepts of a line are respectively a and b, the coordinates of the points of intersection of the line and the axes are

$(a,0)$ and $(0,b)$. The equation of the line through these two points is, therefore,

$$\frac{y-b}{x-0} = \frac{b-0}{0-a}, \qquad \text{or} \qquad \frac{y-b}{x} = -\frac{b}{a}$$

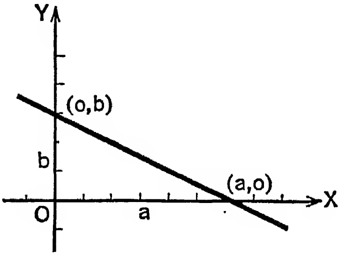


FIG. 36

This may be reduced to

$$bx + ay = ab,$$

and by dividing both sides of the equation by ab, we get

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (13)$$

which is the *intercept form* of the equation of a line.

EXAMPLE. Change the equation $4x - 5y - 8 = 0$ to the intercept form.

By transposing the constant term, and dividing the equation by it, we obtain $\frac{x}{2} - \frac{5y}{8} = 1$. Expressing this in the intercept form, we have $\frac{x}{2} + \frac{y}{-\frac{8}{5}} = 1$.

Another method is to determine the intercepts directly from the equation and substitute them in the intercept form. In this example, the equation has the x-intercept $a = 2$, and the y-intercept $b = -\frac{8}{5}$. Substituting these values in the general intercept form, we obtain the same result as given above.

EXERCISES

Write the equations of the lines which pass through the following points with the indicated slopes.

1. $(-3,2)$, $m = \frac{2}{3}$.

2. $(-2,7)$, $m = -\frac{5}{2}$.

3. $(2,4)$, $m = 3$.

4. $(3,-4)$, $m = \frac{4}{3}$.

5. $(-4,-6)$, $m =$

6. $(6,-2)$, $m = -2$.

7. $(7,-9)$, $m = 4$.

8. $(-2,2)$, $m = 0$.

Write the equations of the lines determined by the following pairs of points.

9. (2,3) and (−3,5).

10. (0,6) and (−2,−3).

11. $(\frac{3}{4},\frac{3}{2})$ and $(\frac{7}{2},3)$.

12. (0,0) and (−3,−5).

13. $(-3\frac{1}{2},5\frac{1}{2})$ and (4,−6).

14. (−8,−1) and (6,−10).

Write the equations of the lines satisfying the following conditions.

15. Passing through the point (1,5) perpendicular to the line determined by the points (−2,−6) and (8,2). Also through (1,5) parallel to the line.

16. Passing through the mid-point of the line segment determined by (−6,20) and (4,−4) and perpendicular to it.

17. Draw the triangle with vertices $A(-2,4)$, $B(1,-1)$ and $C(6,2)$, and find the following:

a. Equations of the sides.
b. Equations of the medians.
c. Equations of the perpendicular bisectors of the sides.
d. Equations of the lines through the vertices parallel to the opposite sides.
e. The angles of the triangle.

18. If $A(8,0)$, $B(6,4)$, and $C(-1,3)$ are the vertices of a triangle, find the equations and angles required in Exercise 17.

Find the slopes and y-intercepts of the following lines:

19. $3x - 5y - 10 = 0$.

20. $x + y + 1 = 0$.

21. $4x + 3y - 18 = 0$.

22. $2x + y = 8$.

23. $3x + y = 7$.

24. $12x + 5y + 50 = 0$.

25. $3y + 2 = 0$.

26. $5x - 7y = 0$.

27. Determine k so that the line $4x + ky - 12 = 0$ shall be (a) parallel and (b) perpendicular to $x = 3y$; to $5x - 3y + 5 = 0$.

28. For each of the following determine a value of k so that the line $2x - 3ky + 8 = 0$ shall pass through the following points: (a) (−4,2); (b) (1,2); (c) (0,2); (d) (3,2).

29. Determine k so that the line $2kx + y - 8 = 0$ shall make a triangle of area 32 square units with the coordinate axes.

30. Write the equation of the line which is perpendicular to the line joining (7,1) with (−2,−2), and which meets this line in the point whose abscissa is −5; in the point whose ordinate is 3.

31. Find the equation of the line through the point (3,5) such that the x-intercept is twice the y-intercept.

32. Find the equation of the line whose y-intercept is -3 and which is perpendicular to the line $2x + 3y - 5 = 0$.

33. Lines pass through the point $(2,\frac{8}{3})$ and form a triangle of area 12 square units with the coordinate axes. What are their equations?

34. A line passes through the intersection of $x + y = 5$ and $2x + 3y = 4$, and through the point (3,5). What is its equation?

35. Find the equation of the line passing through the intersection of $5x - 3y = 1$ and $x + y - 5 = 0$ with a slope of $\frac{2}{3}$.

36. A point moves so that it is always equidistant from the two points $(-2,5)$ and $(5,-2)$. Find its equation in two ways.

37. The coordinates of the vertices of a triangle are $(-3,-4)$, (1,6), and $(5,-2)$. Find the coordinates of the center and the radius of the circumscribed circle.

24. The General Equation of a Line. The most general form of the equation of the first degree in the variables x and y is

$$Ax + By + C = 0, \tag{14}$$

where A, B, and C are any constants, including zero, but with the restriction that A and B cannot be zero simultaneously. We shall now prove the theorem: *Every equation of the first degree in x and y is the equation of a straight line, and conversely.*

If $B \neq 0$ the equation may be solved for y, and written

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is in the form $y = mx + b$ and therefore represents a straight line with slope $m = -\frac{A}{B}$ and y-intercept $b = -\frac{C}{B}$.

If $B = 0$, we cannot solve for y but the equation may be expressed in the form $x = -\frac{C}{A}$ and represents a straight line parallel to the y-axis.

Since the theorem is true for all possible cases, we may say

that every equation of the first degree is the equation of a straight line.

To prove the converse of the theorem, we observe that every straight line in the plane can be expressed either by the equation $x = a$, where a is a constant, or by $y = mx + b$, since every line must either be parallel to the y-axis or intersect it. These equations being first degree, we have proved that the equation of every straight line is a first degree equation.

25. The Normal Form of the Equation of a Line. In the previous articles, we have used the facts from elementary geometry that a straight line is determined by two points, or by one point and a direction. For use in analytic geometry, it is necessary to develop the equation of a line based upon another idea. A straight line is determined if we know the value of the perpendicular distance from the origin to the line and the angle this perpendicular makes with the positive x-axis. The equation of a line expressed in terms of these quantities is called the *normal form* of the equation of a line.

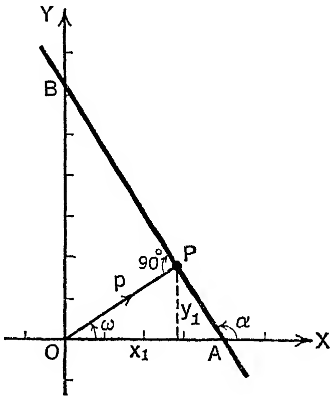


FIG. 37

In Fig. 37, let $OP = p$ be the perpendicular drawn from the origin to the line AB, and let ω represent the angle which OP makes with the positive x-axis. The direction from O to P is assumed positive. Then the coordinates of the point P, lying on AB, are

$$= p \cos \omega \qquad \text{and} \qquad y_1 = p \sin \omega.$$

If α represents the inclination of AB, we have for its slope $m = \tan \alpha = -\cot \omega$, since $\alpha = 90° + \omega$. Then, by using the point-slope form, the equation of the line AB is

$$y - p \sin \omega = -\cot \omega (x - p \cos \omega).$$

By using the identities $\cot \omega = \dfrac{\cos \omega}{\sin \omega}$ and $\sin^2 \omega + \cos^2 \omega = 1$, this equation will reduce to

$$x \cos \omega + y \sin \omega = p, \tag{15}$$

the *normal form* of the equation of a line.

26. Reduction of an Equation to the Normal Form. It often becomes necessary to change the equation of a straight line to the normal form. In order for the general equation

$$Ax + By + C = 0$$

and the normal form

$$x \cos \omega + y \sin \omega - p = 0$$

to represent the same line, the coefficients of the equations must be proportional. Therefore $\dfrac{\cos \omega}{A} = \dfrac{\sin \omega}{B} = \dfrac{-p}{C} = r$, where r is the common ratio. This gives $\cos \omega = rA$, $\sin \omega = rB$, and $p = -rC$, and therefore $\cos^2 \omega + \sin^2 \omega = 1 = r^2A^2 + r^2B^2$. Solving for r, we have $r = \dfrac{1}{\pm\sqrt{A^2 + B^2}}$, and by substitution we get $\cos \omega = \dfrac{A}{\pm\sqrt{A^2 + B^2}}$, $\sin \omega = \dfrac{B}{\pm\sqrt{A^2 + B^2}}$, and $p = \dfrac{-C}{\pm\sqrt{A^2 + B^2}}$ The normal form $x \cos \omega + y \sin \omega - p = 0$ may now be written

$$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} \quad 0,$$

or more compactly

$$\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}} = 0.$$

In order for p to be positive, it is necessary to choose the sign of the radical opposite to that of C, since $p = -rC$.

In the special case when the line passes through the origin we have $p = 0$, since $C = 0$, and we may choose the sign of the radical to be the same as that of B, and thus have ω always less than 180°.

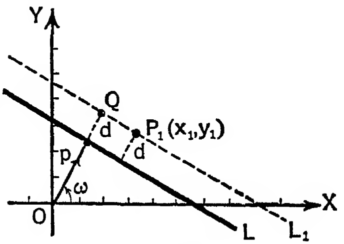


FIG. 38

27. The Distance from a Line to a Point. Perhaps the most important use of the normal form is in finding the perpendicular distance from a line to a point. In Fig. 38, let L be any line in the plane with equation $x \cos \omega + y \sin \omega - p = 0$, and let P_1 be any point with coordinates (x_1,y_1). To find a formula for the distance from the line L to the point P_1, let us draw a line L_1 through P_1 parallel to L. Then the equation of the line L_1

is
$$x \cos \omega + y \sin \omega - p_1 = 0,$$

where $OQ = p_1$. Since the point (x_1,y_1) lies on L_1, it must satisfy the equation of L_1 and therefore

$$x_1 \cos \omega + y_1 \sin \omega - p_1 = 0.$$

From the figure, $OQ = p_1 = p + d$, and substituting this value, we

obtain
$$x_1 \cos \omega + y_1 \sin \omega - p - d = 0.$$

Therefore

$$d = x_1 \cos \omega + y_1 \sin \omega - p = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}. \qquad (16)$$

It should be observed that this formula may be obtained by merely substituting x_1,y_1 for x,y in the normal form of the straight line. This important result may be summarized in the follow-

ing rule: *To find the distance from the line* $Ax + By + C = 0$ *to the point* (x_1,y_1), *change the equation of the line to the normal form by dividing through by* $\pm\sqrt{A^2 + B^2}$, *choose the sign of the radical opposite to the sign of* C, *and substitute the coordinates* (x_1,y_1) *of the given point.* The result is the numerical distance from the line to the point. The algebraic sign will indicate that the point and the origin are on the same or opposite sides of the line, for the sign is negative when the origin and the point are on the same side of the line, and positive when they are on opposite sides. If the line goes through the origin, $C = 0$, and, by choosing the sign of the radical the same as the sign of B, the positive direction from the line will be upward.

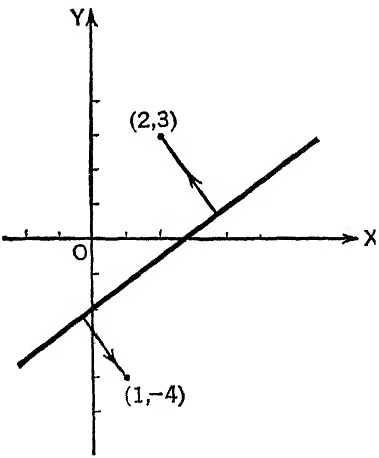


FIG. 39

EXAMPLE. Find the distance from the line

$$3x - 4y - 8 = 0$$

to the point (2,3); to the point (1,−4). In each case give the position of the point with respect to the origin and the line.

Changing $3x - 4y - 8 = 0$ to the normal form, we get

$$\frac{3x - 4y - 8}{+\sqrt{9 + 16}}$$

or $3x - 4y - \quad = 0$. Substituting $x = 2$ and $y = 3$, we have for the distance $d = \dfrac{3(2) - 4(3) - 8}{5} = -\dfrac{14}{5}$. The sign of d indicates that the origin and the point (2,3) are on same side of the line. Similarly, for the distance to the point (1,−4) we get $d = \dfrac{3(1) - 4(-4) - 8}{5} = +\dfrac{11}{5}$, and

the positive sign indicates that this point and the origin are on opposite sides of the line. These results should be compared with the figure.

EXERCISES

Find the equation and draw the lines in each of the following cases.

1. $\omega = 30°$, $p = 6$.
2. $\omega = 45°$, $p = 4$.
3. $\omega = 225°$, $p = 4$.
4. $\omega = 210°$, $p = 3$.
5. $\omega = 150°$, $p = 5$.
6. $\omega = 240°$, $p = 2$.

7. Find the equation of the line which passes through the point $(3,-6)$ with $\omega = 120°$; also through the same point with $p = 3\sqrt{5}$.

Transform each of the following equations to the normal form, draw the figure and indicate on it the values of p and ω.

8. $4x + 3y - 24 = 0$.
9. $15x - 8y + 30 = 0$.
10. $x + 2y = 10$.
11. $12x + 5y = 24$.
12. $4x - y + 16 = 0$.
13. $x - y - 3\sqrt{2} = 0$.
14. $4x - 3y = 0$.
15. $x + y + 5\sqrt{2} = 0$.
16. $5x - 12y - 26 = 0$.
17. $x + y = 0$.

Find the distance from each line to the indicated point, and give the position of the point with respect to the line and to the origin.

18. $7x + 24y - 21 = 0$, $(4,5)$.
19. $x - 4y + 4 = 0$, $(2,1)$.
20. $4x - 3y + 12 = 0$, $(-3,5)$.
21. $2x - 3y - 2 = 0$, $(4,0)$.
22. $x + 2y = 0$, $(-2,3)$.
23. $3x - 5y = 0$, $(4,-2)$.
24. $2y - 1 = 0$, $(-4,3)$.
25. $3x - 8 = 0$, $(3,5)$.

28. The Equations of the Angle Bisectors. Let it be required to find the equations of the lines which bisect the angles made by two intersecting lines. From plane geometry we know that an angle bisector is a line equidistant from the two sides of the angle. Let the equations of the two lines which form the angle be

$$L_1 \equiv a_1x + b_1y + c_1 = 0,$$

and

$$L_2 \equiv a_2x + b_2y + c_2 = 0.$$

Consider a point P_1 (Fig. 40) on the desired bisector and represent the perpendicular distances from the two given lines to this point by d_1 and d_2.

Now
$$d_1 = \frac{a_1x_1 + b_1y_1 + c_1}{\pm\sqrt{a_1^2 + b_1^2}},$$

and
$$d_2 = \frac{a_2x_1 + b_2y_1 + c_2}{\pm\sqrt{a_2^2 + b_2^2}}.$$

Hence
$$d_1 = \pm d_2,$$

or
$$\frac{a_1x_1 + b_1y_1 + c_1}{\pm\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x_1 + b_2y_1 + c_2}{\pm\sqrt{a_2^2 + b_2^2}}$$

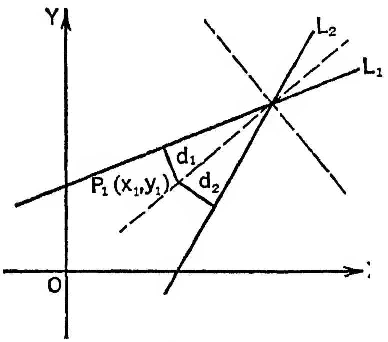


FIG. 40

will give the equations of both angle bisectors. To get a particular one in a numerical problem is a simple matter. The signs of the perpendicular distances may be determined from the figure to give the desired bisector, or we can compute both bisectors and select the desired one by comparing slopes or intercepts. We shall illustrate the method by finding the equations of both angle bisectors for a definite pair of lines.

EXAMPLE 1. Find the equations of the bisectors of the angles formed by the lines $5x - 12y + 10 = 0$ and $12x - 5y + 15 = 0$.

Let $P(x,y)$ be on the bisector of the angle indicated in Fig. 41.

Then $d_1 = \dfrac{5x - 12y + 10}{-13}$ and $d_2 = \dfrac{12x - 5y + 15}{-13}$.

To obtain both bisectors we write

$$d_1 = \pm d_2.$$

Therefore the equations of both bisectors are given by

$$\frac{5x - 12y + 10}{13} = \pm\frac{12x - 5y + 15}{-13},$$

or $$5x - 12y + 10 = \pm(12x - 5y + 15).$$

Using the positive sign, we obtain

$$7x + 7y + 5 = 0, \quad \text{(a)}$$

and using the negative sign, we have

$$17x - 17y + 25 = 0. \quad \text{(b)}$$

These two bisectors can be distinguished by computing their y-intercepts.

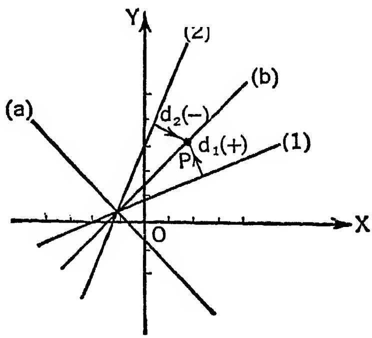


FIG. 41

If we desire to find only one bisector, say (b), the figure shows that d_1 is positive since the point P and the origin are on opposite sides of line (1), and that d_2 is negative since the point P and the origin are on the same side of line (2). Therefore $d_1 = -d_2$ will give the equation $17x - 17y + 25 = 0$ as the bisector of the acute angle. Each line should be carefully checked with the figure.

EXAMPLE 2. Find the equations of the angle bisectors of the triangle whose sides are $4x - 3y = 0$, $3x + 4y + 8 = 0$, and $x - 5 = 0$, and prove that they meet in a common point.

We first find the distance from each line to a point (x,y) on the bisector, that is,

$$d_1 = \frac{4x - 3y}{-5}, \qquad d_2 = \frac{3x + 4y + 8}{-5}, \qquad d_3 = \frac{x + 0 \cdot y - 5}{1}.$$

We observe from Fig. 42 that $d_1 = \pm d_2$ will give the equations of both bisectors of the angles at the vertex B. However, in

order to obtain the bisector of the interior angle, we note that d_1 is negative since it lies below a line which passes through the origin, and d_2 is also negative since the point P and the origin are on the same side of the line from which d_2 is measured. Therefore using $-d_1 = -d_2$, we have

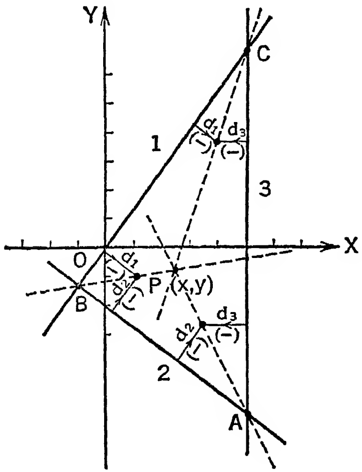


FIG. 42

$$\frac{4x - 3y}{5} \qquad 3x + 4y + 8$$

This gives $x - 7y - 8 = 0$ as the equation of the bisector of the interior angle at B. This line should be checked by computing its intercepts and comparing them with the figure.

In a similar manner $d_2 = \pm d_3$ will give both bisectors at A, but to get the bisector of the interior angle, we notice that both d_2 and d_3 have the same sign and therefore we use $d_2 = +d_3$. This gives $\dfrac{3x + 4y + 8}{-5} = \dfrac{x - 5}{1}$, which reduces to $8x + 4y - 17 = 0$, as the desired equation of the angle bisector. We can check this equation by noting that it has a negative slope as the figure requires.

Finally $d_1 = \pm d_3$ will give both bisectors at C, but $-d_1 = -d_3$ will give the bisector of the interior angle. Therefore

$$\frac{4x - 3y}{-5} = \frac{x - 5}{1}, \qquad \text{or} \qquad 9x - 3y - 25 = 0$$

is the desired equation.

To prove that the three bisectors meet in a point, we show that the equations of the bisectors have a common algebraic solution. This may be done by solving any two of them together

to get the coordinates of their common point, and then showing that these coordinates satisfy the third equation. Solving $x - 7y = 8$ and $8x + 4y = 17$ simultaneously, we get $x = \frac{151}{60}$ and $y = -\frac{47}{60}$. These values will satisfy the third equation $9x - 3y = 25$ upon substitution. Therefore the three lines meet in the point $\left(\frac{151}{60}, -\frac{47}{60}\right)$. From plane geometry we know that the bisectors of the angle of a triangle intersect in a point which is the center of the inscribed circle.

29. The Area of a Triangle. Let us find the area of a general triangle with vertices $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x_3,y_3)$. The base $AB = b = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Fig. 43). To find a formula for the altitude h, we first find the equation of AB, using the two-point form of a straight line. This gives for AB

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which may be written

$$-(y_2 - y_1)x + (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0.$$

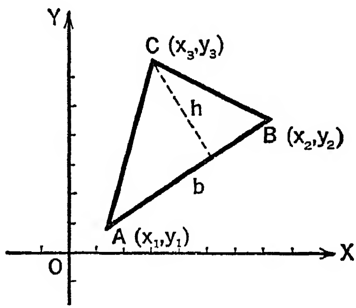


FIG. 43

The formula for the distance from this line to the vertex C is

$$h = \frac{-(y_2 - y_1)x_3 + (x_2 - x_1)y_3 + (x_1y_2 - x_2y_1)}{\pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Then the area of the triangle ABC, that is, $K = \frac{1}{2}bh$, becomes

$$K = \pm\tfrac{1}{2}[-(y_2 - y_1)x_3 + (x_2 - x_1)y_3 + (x_1y_2 - x_2y_1)$$

This expression may be expanded and its terms rearranged in the form:

$$K = \pm\tfrac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)].$$

This form is conveniently expressed in determinant notation by

$$K = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{17}$$

The $\pm$ sign should be so chosen as to give a positive value to the area. If the vertices are taken in sequence so as to keep the area to the left, K will be a positive quantity.

EXERCISES

Find the equations of the bisectors of the angles formed by the given lines. Draw a figure to illustrate each exercise.

1. $x - 3y + 6 = 0$ and $6x - 2y + 3 = 0$.
2. $4x - 3y + 17 = 0$ and $3x - 4y + 7 = 0$.
3. $7x - 6y + 84 = 0$ and $6x + 7y - 42 = 0$.
4. $x + y + 4 = 0$ and $x - y + 8 = 0$.
5. $3x + 4y + 6 = 0$ and $8x - 15y + 16 = 0$.

Find the equations of the bisectors of the interior angles of the triangles whose sides are given and prove that these bisectors meet in a point.

6. $x + y - 5 = 0$, $x - y + 15 = 0$, and $y = 0$.
7. $3x - 4y + 22 = 0$, $5x + 12y + 10 = 0$, and $15x + 8y + 30 = 0$.
8. $x + y - 15 = 0$, $x - 7y - 11 = 0$, and $17x + 7y + 65 = 0$.
9. $3x + 4y - 10 = 0$, $8x - 6y + 3 = 0$, and $12x + 5y - 15 = 0$.
10. $3x - 4y = 0$, $4x + 3y - 50 = 0$, and $y = 0$.
11. $x - 3y + 15 = 0$, $3x + y + 15 = 0$, and $3x - y + 6 = 0$.
12. $5x - 12y - 3 = 0$, $12x + 5y + 24 = 0$, and $5x + 12y - 75 = 0$.

Find the center and radius of the circle inscribed in the triangle whose sides are:

13. $x - 2y - 4 = 0$, $2x + y - 12 = 0$, and $2x - y + 4 = 0$.
14. $3x + 4y - 12 = 0$, $4x - 3y + 9 = 0$, and $8x - 15y - 54 = 0$.

Find the areas of the triangles with the following vertices:

15. $(0,0)$, $(-3,4)$, and $(3,5)$.
16. $(3,-2)$, $(-1,3)$, and $(6,5)$.
17. $(7,5)$, $(-11,6)$, and $(3,-2)$.
18. $(-2,2)$, $(4,2)$, and $(1,5)$.

30. Systems of Lines. An equation of the first degree in x and y which contains an arbitrary constant—that is, a constant to which any value may be assigned—may be considered as representing an infinite number of straight lines, since infinitely many values may be given to the constant and each value will determine a definite line. Such an equation is said to represent a system of lines and the arbitrary constant is called a **parameter.**

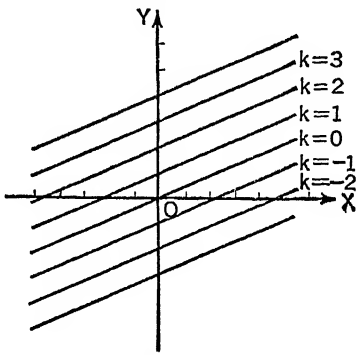


Fig. 44

For example, if the equation is $y = \frac{1}{2}x + k$ and we give to k different values, we get a series of lines with slope $\frac{1}{2}$. For

$$\begin{aligned} k &= 0, & y &= \tfrac{1}{2}x; \\ k &= 1, & y &= \tfrac{1}{2}x + 1; \\ k &= 2, & y &= \tfrac{1}{2}x + 2; \\ k &= -1, & y &= \tfrac{1}{2}x - 1; \text{ etc.} \end{aligned}$$

Fig. 44 shows a few members of this family, or system, of lines. This system is characterized by the fact that all members have the same slope.

Similarly, if the equation is $y - 2 = k(x - 2)$, we have for

$$\begin{aligned}
k &= 0, & y - 2 &= 0;\\
k &= 1, & y - 2 &= x - 2;\\
k &= 2, & y - 2 &= 2(x - 2);\\
k &= -1, & y - 2 &= -(x - 2);\ \text{etc.}
\end{aligned}$$

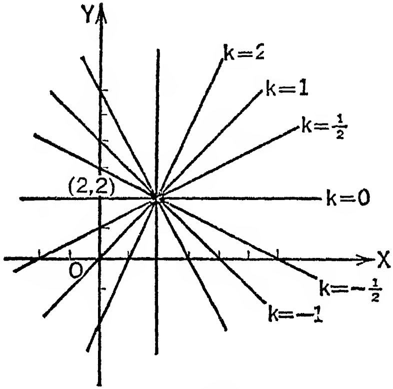


FIG. 45

This represents a system of lines passing through the point (2,2) as shown in Fig. 45.

EXAMPLE 1. Find the equation of the system of lines having a y-intercept of 2 units.

Here we have just one point of the line given. The equation of any line with y-intercept 2 is $y = kx + 2$. For each value of the parameter k we have a line of different slope, but all the lines have the same y-intercept.

EXAMPLE 2. Write down the equation of the system of lines parallel to the line $2x - 3y + 6 = 0$, and then determine the line of the system which passes through the point (4,3).

Any line parallel to the given line may be written $2x - 3y = k$. We notice that for each value of k this equation represents a line with slope $\frac{2}{3}$, and therefore we have a system

of lines parallel to the given line with k as the parameter. If we desire to determine a line of the system which passes through the point (4,3), we substitute these coordinates in the equation of the system and so determine a value for k, that is, $2(4) - 3(3) = k$, or $k = -1$. Therefore $2x - 3y + 1 = 0$ is the desired equation.

EXAMPLE 3. Find the equation of the line which is perpendicular to $4x + 3y + 6 = 0$ and which passes through the point $(-2,3)$.

First let us write down the equation of all the lines perpendicular to the given line. This is $3x - 4y = k$. By comparing slopes it can be shown that all these lines are perpendicular to the given line. Next we select just one line of the system which passes through $(-2,3)$ by substituting these values in the equation of the system and getting $3(-2) - 4(3) = k$, or $k = -18$. Therefore the equation of the line is $3x - 4y + 18 = 0$.

It is important to note that the equation of the system of lines parallel to $ax + by + c = 0$ has the form $ax + by = k$, and the system perpendicular to $ax + by + c = 0$ has the form $bx - ay = k$. This makes it very easy to write down the equation of all lines that are parallel or perpendicular to a given line.

31. Lines through the Intersection of Two Given Lines. Consider the two lines

$$L_1 \equiv a_1x + b_1y + c_1 = 0,$$

and

$$L_2 \equiv a_2x + b_2y + c_2 = 0.$$

Next consider the equation

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0,$$

or briefly

$$L_1 + kL_2 = 0.$$

This equation represents a straight line since it is clearly first degree in x and y. Since it contains an arbitrary constant, or

parameter, k it represents a system of lines. If (x_1, y_1) is the point of intersection of the two original lines, we know that $L_1 = a_1x_1 + b_1y_1 + c_1 = 0$ and $L_2 = a_2x_1 + b_2y_1 + c_2 = 0$ since the point lies on each line. Also, this point satisfies the equation of the system of lines $L_1 + kL_2 = 0$ since it satisfies $L_1 = 0$ and $L_2 = 0$ separately. This is equivalent to saying that all the lines determined by the equation $L_1 + kL_2 = 0$ pass through the point of intersection of $L_1 = 0$ and $L_2 = 0$. By using this equation, we can quite often solve exercises involving intersecting lines without actually finding their points of intersection. Let us illustrate the method by some examples.

Example 1. Find the equation of the line which passes through the intersection of the lines $x - 2y + 4 = 0$ and $2x + y + 6 = 0$, and through the point $(3, -2)$.

The system of lines through the intersection of the given lines is given by the equation $(x - 2y + 4) + k(2x + y + 6) = 0$. To select the line of the system which passes through the point $(3, -2)$, we substitute $x = 3$, $y = -2$ and get

$$(3 + 4 + 4) + k(6 - 2 + 6) = 0,$$

which reduces to $10k = -11$. Solving for k and substituting its value in the equation of the system, we have

$$(x - 2y + 4) - \tfrac{11}{10}(2x + y + 6) = 0,$$

or

$$12x + 31y + 26 = 0.$$

Example 2. Find the equation of the line of slope $\frac{4}{3}$ which passes through the point of intersection of the lines $4x - 5y - 5 = 0$ and $2x + 3y - 11 = 0$.

The system of lines through the intersection of the given lines has the equation $(4x - 5y - 5) + k(2x + 3y - 11) = 0$. This equation may be rearranged as

$$(4 + 2k)x + (3k - 5)y - 11k - 5 = 0,$$

and, by solving for y, it takes the form $y = mx + b$, or

$$y = \left(\frac{4 + 2k}{5 - 3k}\right)x + \frac{11k + 5}{3k - 5}.$$

Hence the slope is $m = \dfrac{4 + 2k}{5 - 3k}$ and as this is to be equal to $\dfrac{4}{3}$, we have $\dfrac{4 + 2k}{5 - 3k} = \dfrac{4}{3}$, and finally $k = \dfrac{4}{9}$. Substituting this value of k, we find the desired equation to be

$$(4x - 5y - 5) + \tfrac{4}{9}(2x + 3y - 11) = 0,$$

or

$$44x - 33y - 89 = 0.$$

EXERCISES

1. Find the equation of the system of lines which pass through the point $(-1,3)$. Determine the particular line of the system which passes through $(-2,1)$.

2. Write the equation of the system of lines with slope $-\frac{2}{3}$, and determine the line of the system which passes through the point $(1,5)$.

3. A system of lines has a slope $-\frac{4}{3}$. What is the equation of the line of the system which makes a triangle of area 6 square units with the coordinate axes?

4. Find the equation of the system of lines parallel to

$$2x - 3y + 8 = 0.$$

Which line of the system passes through the point $(4,2)$?

5. Find the particular line of the system of lines passing through the intersection of $3x - 2y - 16 = 0$ and $x + 3y - 2 = 0$ which has a slope $\frac{2}{5}$.

6. Find the equation of the system of lines through the intersection of $2x - 3y - 16 = 0$ and $2x + 6y - 1 = 0$. What is the equation of the line of the system which passes through $(6,4)$?

7. A line passes through the point of intersection of $x - 2y + 7 = 0$ and $4x + 3y - 9 = 0$ and through the point $(5,-3)$. Find its equation.

8. Find the equation of the line which passes through the intersection of the two lines $5x + 10y + 11 = 0$ and $2x + y + 14 = 0$ and which is perpendicular to $7x + y + 1 = 0$.

9. Find the line which passes through the intersection of the lines $3x - y - 4 = 0$ and $5x - y + 2 = 0$, and which is parallel to $3x + 2y - 12 = 0$.

10. Write down the equation of the system of lines passing through the point (3,4), and then determine the particular line of the system which is 5 units distant from the origin by using the normal form of a straight line.

Given a triangle with vertices $A(3,4)$, $B(-1,1)$, and $C(5,2)$, write down the system of lines through the vertex A, and then determine the following:

11. The equation of the median through A.

12. The equation of the altitude through A.

13. The equation of the line passing through A which is parallel to the side BC.

Draw a general triangle with vertices $(2a,0)$, $(2b,0)$, and $(0,2c)$, and then solve the following exercises, using any method.

14. Find the equations of the medians and prove that they meet in a point.

15. Find the equations of the altitudes and prove that they meet in a point.

16. Find the equations of the perpendicular bisectors of the sides and prove that they meet in a point.

17. Prove that the three points found in Exercises 14, 15 and 16 lie on one line, which is known as *Euler's line.*

CHAPTER IV

SPECIAL EQUATIONS OF THE SECOND DEGREE

32. Introduction. We shall approach the study of equations of the second degree from the point of view of finding *the equation of a locus.* That is, the law governing the motion of a point in a plane will be given as the definition of a curve, and from this definition we shall find the algebraic expression which describes the path traced by the moving point. As this statement indicates and as has been the case heretofore, all points, lines, et cetera, used in a definition are in the same plane. For this reason the curves to be discussed, namely, the **circle,** the **parabola,** the **ellipse** and the **hyperbola,** are called *plane curves.*

THE CIRCLE

33. Definition and Standard Equation. *A circle is the locus of a point which moves so that its distance from a fixed point is always constant.* The fixed point is called the **center,** and the constant distance the **radius,** of the circle. To determine its equation consider Fig. 46.

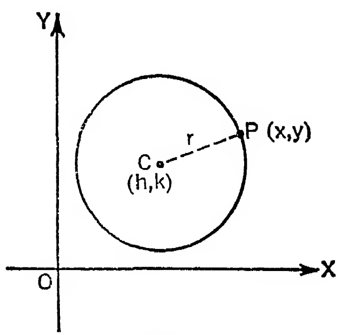


Fig. 46

Let $C(h,k)$ be the fixed point, $P(x,y)$ the moving or tracing point, and $CP = r$ the constant distance. Hence by means of the distance formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r,$$

or
$$(x - h)^2 + (y - k)^2 = r^2. \qquad (18)$$

Since this equation is satisfied by all points on the circle and by no other points, it is called *the equation of a circle with center* (h,k) *and radius* r.

If the center is at the origin, $h = k = 0$ and our equation becomes

34. The General Equation. We have just seen that

$$(x - h)^2 + (y - k)^2 = r^2$$

is the equation of a circle with center (h,k) and radius r. By expanding the binomial terms and transposing r^2, this expression becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

But this is of the form

$$+ y^2 + Dx + Ey + F = 0, \qquad (19)$$

if we make the substitutions $D = -2h$, $E = -2k$, $F = h^2 + k^2 - r^2$. Hence, we may say that it is always possible to write the equation of a circle in the form (19).

To show the converse, that is, that every equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

represents a circle, we proceed as follows. Transpose the constant term F and complete the square on each set of terms $x^2 + Dx$ and $y^2 + Ey$, thereby obtaining

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F,$$

where $\frac{D^2}{4}$ and $\frac{E^2}{4}$ have been added to the right member to preserve the equality. An equivalent form of this equation is

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{1}{4}(D^2 + E^2 - 4F),$$

which is the algebraic expression of the condition that a tracing point (x,y) remain at a constant distance $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$ from a fixed point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$. Hence, it is the equation of a circle with center $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

Since the radius is expressed in terms of a radical, the following cases may arise:

(a) When $D^2 + E^2 - 4F < 0$, the radius is imaginary and there is no real locus. The circle is called *imaginary*.

(b) When $D^2 + E^2 - 4F = 0$, the radius is zero and the circle shrinks to a point, the center. It is sometimes called a *point circle*.

(c) When $D^2 + E^2 - 4F > 0$, the radius is real and we have a real circle.

Since $ax^2 + ay^2 + bx + cy + d = 0$, $(a \neq 0)$ can be reduced to the form (19) by dividing through by a and substituting $D = \frac{b}{a}$, $E = \frac{c}{a}$, $F = \frac{d}{a}$, we may say that *every equation of the second degree in x and y, in which the xy term is missing and the coefficients of the x^2 and y^2 terms are the same, is the equation of a circle.* Inasmuch as the symbols selected to represent variables is a matter of choice, this statement is true when x and y are replaced by other letters.

EXAMPLE. Find the center and radius of the circle $4x^2 + 4y^2 - 12x + 4y - 26 = 0$, and draw the figure.

Dividing through by 4 in order to reduce the equation to the general form (19), we obtain

$$x^2 + y^2 - 3x + y - \tfrac{13}{2} = 0.$$

Therefore,

$$D = -3,\ E = 1,\ F = -\frac{13}{2}, \qquad \text{and} \qquad -\frac{D}{2} = \frac{3}{2},\ -\frac{E}{2} = -\frac{1}{2},$$

$r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F} = \frac{1}{2}\sqrt{9 + 1 + 26} = 3$. Fig. 47 shows the circle with center $(\frac{3}{2}, -\frac{1}{2})$ and $r = 3$.

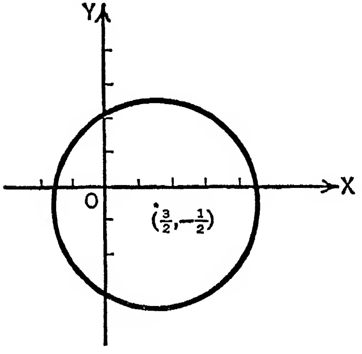


FIG. 47

The problem may be solved directly, without the necessity of remembering the formulas for center and radius, by completing the squares on the x and y terms. After dividing through by 4, the equation may be written

$$x^2 + y^2 - 3x + y = \tfrac{13}{2}.$$

Hence, by completing the squares,

$$x^2 - 3x + \tfrac{9}{4} + y^2 + y + \tfrac{1}{4} = \tfrac{13}{2} + \tfrac{9}{4} + \tfrac{1}{4} = 9,$$

or

$$(x - \tfrac{3}{2})^2 + (y + \tfrac{1}{2})^2 = 9.$$

By comparison with

$$(x - h)^2 + (y - k)^2 = r^2,$$

we see that the center is at $(\frac{3}{2}, -\frac{1}{2})$ and the radius is $r = 3$.

EXERCISES

Write the equations of the following circles and draw the figures.

1. Center at (0,0); $r = 4$.

2. Center at (2,4); $r = 8$.

3. Center at (2,−2); $r = 6$.

4. Center at (0,4); $r = 1$.

5. Center at $(-4,2)$ and touching the y-axis.

6. Center at $(2,3)$ and touching the x-axis.

7. Center at $(-2,2)$ and touching both axes.

8. Center at $(3,2)$ and touching the line $x + y + 1 = 0$.

9. Center at $(-3,5)$ and passing through the point $(3,-3)$.

10. Center at (a,b) and passing through the origin.

Determine the center and radius of each of the following circles; draw the curve when possible.

11. $x^2 + y^2 - 2x + 4y - 11 = 0$. 12. $x^2 + y^2 + 8x = 0$.

13. $4x^2 + 4y^2 - 4x + 8y + 5 = 0$.

14. $x^2 + y^2 - 6x - y + 18 = 0$. 15. $3x^2 + 3y^2 + 8x + 4y = 0$.

16. $3x^2 + 3y^2 - 5x + 2y + 1 = 0$.

17. A point moves in such a way that its perpendicular distance from the line $x = 4$ is always equal to the square of its distance from the origin. Find the equation of the locus.

18. A circle with center at $(6,4)$ has a radius equal to the segment of the line $3x - 4y - 24 = 0$ which is intercepted by the coordinate axes. What is its equation?

19. Find the equation of the straight line joining the centers of the circles $x^2 + y^2 - 2x + 3y - 4 = 0$ and $2x^2 + 2y^2 + 4x - 8y - 5 = 0$.

20. A square, with vertices $(a,0)$, $(0,a)$, $(-a,0)$ and $(0,-a)$ is inscribed in a circle. What is the equation of the circle?

21. What is the equation of the circle inscribed in the square of Exercise 20?

22. Show analytically that every angle inscribed in a semicircle is a right angle.

Prove that the locus in each of the following problems is a circle.

23. A point moves in such a way that the square of its perpendicular distance from the base of an isosceles triangle is equal to the product of its perpendicular distances from the remaining two sides.

24. A point moves in such a way that the sum of the squares of its perpendicular distances from the three sides of an equilateral triangle is constant.

25. A point moves in such a way that the square of its distance from a fixed point is proportional to its perpendicular distance from a fixed straight line.

35. Circle Determined by Three Conditions. By examining the standard forms of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$ and $x^2 + y^2 + Dx + Ey + F = 0$, we see that each contains three arbitrary constants. Hence, in order to obtain the equation of a particular circle, we must be able to set up three independent equations from which the values of these constants, h, k, r or D, E, F, can be found. Such equations are the analytical expressions of conditions which the circle must satisfy, and since in general three such conditions will lead to three independent equations, we speak of a circle as being determined by three conditions. While it is often true that the given conditions determine just one circle, as for instance, "three points not in the same straight line determine one and only one circle," this is not always the case, because it may happen that several circles satisfy the same conditions. For example, four distinct circles may be drawn touching three given lines.

The usual method of solving problems of the type considered here is to decide which of the standard forms of the equation of a circle is to be used and then set up the three independent equations in the constants involved. It is often the case, however, that an ingenious student will devise a shorter and more satisfactory method of attack by considering the geometry involved in a particular problem.

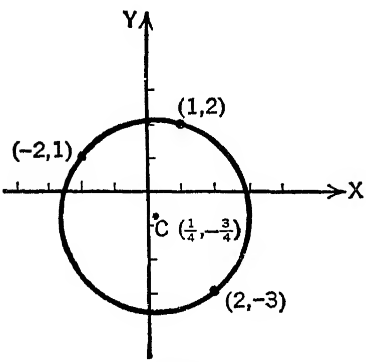


FIG. 48

EXAMPLE 1. Find the equation of the circle through the points (1,2), (−2,1) and (2,−3).

Let the standard form
$$x^2 + y^2 + Dx + Ey + F = 0$$
represent the circle. Then, since each point is on the circle, the coordinates of the given points must satisfy the equation.

Making the substitutions, we obtain the three equations

$$\begin{aligned} 1 + 4 + D + 2E + F &= 0 \\ 4 + 1 - 2D + E + F &= 0 \\ 4 + 9 + 2D - 3E + F &= 0, \end{aligned}$$

which are solved for D, E and F. The values are found to be

$$D = -\tfrac{1}{2},\ E = \tfrac{3}{2},\ F = -\tfrac{15}{2}.$$

Substituting these values in the general equation, we have

$$2x^2 + 2y^2 - x + 3y - 15 = 0,$$

which is the equation of a circle (Fig. 48) through the points (1,2), (−2, 1) and (2,−3), with center at $(\frac{1}{4}, -\frac{3}{4})$ and $r = \frac{1}{4}\sqrt{130}$.

The accuracy of the work should be checked by substituting the coordinates of the given points in the final equation.

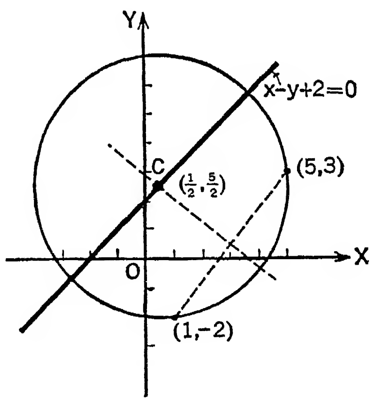


FIG. 49

EXAMPLE 2. Find the equation of the circle (Fig. 49) passing through the points (1,−2) and (5,3) and having its center on the line $x - y + 2 = 0$.

Choosing

$$(x - h)^2 + (y - k)^2 = r^2$$

as the standard form, we substitute the given points and obtain

$$\begin{aligned} (1 - h)^2 + (-2 - k)^2 &= r^2 \\ (5 - h)^2 + (3 - k)^2 &= r^2 \end{aligned}$$

as two of the equations in h, k, and r. The third is found by substituting the coordinates of the center (h,k) in the equation

of the line $x - y + 2 = 0$, since this line passes through the center. We have

$$h - k + 2 = 0.$$

Solving the three equations simultaneously for h, k and r, we find

$$h = \tfrac{1}{2},\ k = \tfrac{5}{2},\ r = \tfrac{1}{2}\sqrt{82}.$$

Hence the required equation of the circle is

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{5}{2})^2 = \tfrac{82}{4}, \quad \text{or} \quad x^2 + y^2 - x - 5y - 14 = 0.$$

An alternative method of solving the problem is as follows. We know from elementary geometry that the center of a circle is the point of intersection of a line through its center and a perpendicular bisector of a chord. The slope of the chord joining the points $(1,-2)$ and $(5,3)$ is $\tfrac{5}{4}$, and the coordinates of the mid-point of this chord are $(3,\tfrac{1}{2})$. Hence the equation of the perpendicular bisector is

$$y - \tfrac{1}{2} = -\tfrac{4}{5}(x - 3), \qquad \text{or} \qquad 8x + 10y - 29 = 0.$$

Solving this equation simultaneously with $x - y + 2 = 0$, the equation of the line on which the center of the circle lies, we find the coordinates of the center of the required circle to be $(\tfrac{1}{2},\tfrac{5}{2})$. By finding the distance between the center and either of the given points $(1,-2)$ or $(5,3)$, we obtain $r = \tfrac{1}{2}\sqrt{82}$. Therefore the equation of the circle is

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{5}{2})^2 = \tfrac{82}{4}, \quad \text{or} \quad x^2 + y^2 - x - 5y - 14 = 0.$$

EXERCISES

Find the equations of the circles satisfying the following conditions. Draw a figure in each case and check your results.

1. Passing through the points $(0,0)$, $(-2,-1)$, $(4,5)$.

2. Passing through the points $(0,4)$, $(0,-4)$, $(6,0)$.

3. Passing through the points $(-1,3)$, $(7,-1)$, $(2,9)$.

4. Passing through the points $(7,4)$, $(5,10)$, $(-9,-4)$.

5. Having intercepts of 2 and 3 on the x and y axis, respectively, and passing through the origin.

6. Passing through the intersection of the lines $x + y = 0$ and $x - y + 4 = 0$, and through the points $(5,3)$, $(2,-4)$.

7. Passing through the points $(5,0)$, $(0,-3)$, and having its center on the line $x - y = 0$.

8. Passing through the points $(1,-1)$, $(-5,2)$, and having its center on the line $x - y + 7 = 0$.

9. Passing through the points $(-3,2)$, $(1,5)$, and having its center on the line $x = 5$.

10. Passing through the points $(2,8)$, $(-3,-4)$, and having its center on the x-axis.

11. Of radius $\sqrt{53}$ and passing through the points $(-5,3)$, $(4,-2)$.

12. Of radius 4, centers on the line $2x + y - 4 = 0$, and passing through the point $(3,2)$.

13. Touching both axes and passing through the point $(6,3)$.

14. Circumscribing the triangle whose vertices are $(-5,0)$, $(0,5)$, $(5,0)$.

15. Circumscribing the triangle whose vertices are $(-2,3)$, $(4,-2)$, $(2,8)$.

16. Circumscribing the triangle whose sides are the lines

$$3x + y - 5 = 0, \qquad 2x - y - 4 = 0 \qquad \text{and} \qquad x + y + 1 = 0.$$

17. Inscribed in the triangle whose sides are

$$3x + 4y + 12 = 0, \qquad 4x - 3y - 12 = 0 \qquad \text{and} \qquad y = 0.$$

18. Inscribed in the triangle of Exercise 14.

19. With centers on the line $x - 3y + 9 = 0$, passing through the origin, and having an area of 25π square units.

20. Passing through the intersections of the circles

$$- 5y = 0 \qquad \text{and} \qquad x^2 + y^2 + x - 4y + 1 = 0,$$

and having its center on the line $2x - y + 3 = 0$.

36. Circles through the Intersections of Two Given Circles. The last problem of the above exercises suggests that we try to set up an equation which will represent a family of circles passing through the intersections of two given circles. To this end, let

$$C_1 \equiv x^2 + y^2 + D_1x + E_1y + F_1 = 0$$

and $$C_2 \equiv x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

be the equations of the given circles. Now write the equation formed by multiplying one of the given equations by a constant k and adding it to the other. For instance, $C_1 + kC_2 = 0$, or written in full,

$$x^2 + y^2 + D_1x + E_1y + F_1 \\ + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0.$$

By collecting like terms, this equation may be put in the form

$$(1 + k)x^2 + (1 + k)y^2 + (D_1 + kD_2)x \\ + (E_1 + kE_2)y + F_1 + kF_2 = 0,$$

which we recognize as the equation of a circle for every value of k except $k = -1$. Since we have a different circle whenever the value of k is changed, we say that $C_1 + kC_2 = 0$ is the equation of a *family of circles*. Furthermore, each circle of this family will pass through the intersections of the given circles, because any pair of values of x and y which satisfy both $C_1 = 0$ and $C_2 = 0$ will likewise satisfy $C_1 + kC_2 = 0$. Therefore, we conclude that *$C_1 + kC_2 = 0$ is the equation of a family of circles through the intersections of $C_1 = 0$ and $C_2 = 0$.* Obviously, the equation of the family may be written in the form $C_2 + kC_1 = 0$. That is, it is immaterial which of the given circles is multiplied by the constant.

Since we have already imposed two conditions on any circle of the family, namely, that it go through the intersections of the given circles, it is necessary to have only one further condition

in order to find the value of k and hence the equation of a particular circle of the family.

EXAMPLE. Find the equation of the circle which passes through the intersections of $C_1 \equiv x^2 + y^2 - 2x - 2y - 2 = 0$ and $C_2 \equiv x^2 + y^2 + 4x - 6y + 4 = 0$, and cuts the y-axis 4 units below the origin.

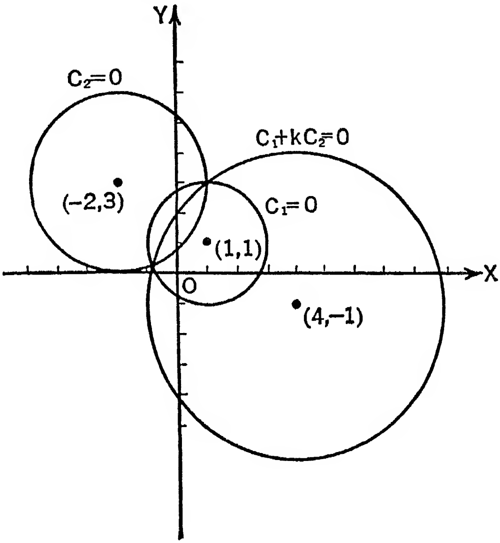


FIG. 50

First, write the equation of the family of circles through the intersections of the given circles. That is,

$$x^2 + y^2 - 2x - 2y - 2 + k(x^2 + y^2 + 4x - 6y + 4) = 0.$$

Now since the particular circle in which we are interested passes through the point $(0,-4)$, we substitute these values of x and y and find

$$44k + 22 = 0, \qquad \text{or} \qquad k = -\tfrac{1}{2}.$$

Substituting this value of k in the equation of the family of circles above and simplifying, we obtain

$$x^2 + y^2 - 8x + 2y - 8 = 0$$

as the desired equation. Fig. 50 shows the three circles.

37. The Radical Axis of Two Circles. If we examine the equation $C_1 + kC_2 = 0$ of Art. 36 for the special case when $k = -1$, we see that it assumes the form $C_1 - C_2 = 0$, or

$$(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0. \tag{20}$$

Since this equation is of the first degree in x and y, it always represents a straight line. It is called the **radical axis** of the two circles. If the circles intersect in real points, the radical axis is called the *common chord;* if the circles are tangent, it is called the *common tangent.* Even though the circles have no real points in common, the radical axis is a real line and may be plotted, provided the circles are not concentric.

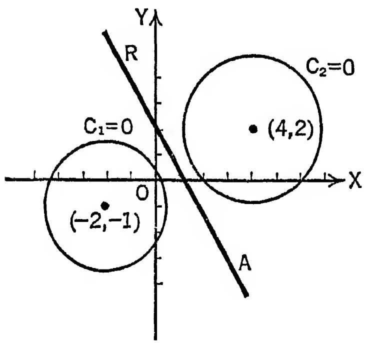


FIG. 51

EXAMPLE. Find the radical axis of the circles

$$x^2 + y^2 + 4x + 2y - 1 = 0$$

and $$x^2 + y^2 - 8x - 4y + 12 = 0.$$

Draw the circles and the radical axis.

Subtracting the second equation from the first, that is, using $C_1 - C_2 = 0$, we have

$$12x + 6y - 13 = 0$$

as the desired equation. Fig. 51 shows the circles and the line.

EXERCISES

Find the equation of the circle which passes through the points of intersection of

$$x^2 + y^2 - 6x - 12y + 41 = 0$$

and

$$x^2 + y^2 - 10x - 8y + 37 = 0$$

and satisfies the further condition stated below. Draw a figure in each case.

1. Passing through the point (0,9).

2. Passing through the point $(-1,2)$.

3. Cutting the x-axis 3 units to the left of the origin.

4. Cutting the y-axis at $y = -5$.

5. Having its center on the x-axis.

6. Having its center on the y-axis.

7. Having its center at (6,3). **8.** Having its center at $(-2,11)$.

9. Having its center on the line $x + 2y + 3 = 0$.

10. With center on the line $x + y = 0$. Give the geometric interpretation of your answer.

Use equations

$$x^2 + y^2 - 2x - y - 15 = 0$$

and

$$x^2 + y^2 + 10x - 9y + 29 = 0$$

in working Exercises 11–16. Draw figures.

11. Find the points of intersection of the circles.

12. Find the equation of the common chord.

13. Find the length of the common chord.

14. Find the equation of the line joining the centers of the given circles. This line is called the **line of centers** of the circles.

15. Show that the center of every circle through the intersections of the given circles lies on the line of centers.

16. By means of slopes show that the common chord of the circles meets the line of centers at right angles.

17. Find the equation of the radical axis of the circles whose equations are $x^2 + y^2 + 4x + 8y + 16 = 0$ and $x^2 + y^2 - 10y + 16 = 0$. Draw a figure.

18. The point of intersection of the radical axes of three circles taken in pairs is called the **radical center**. Find this point for the

circles $x^2 + y^2 + 4x + 3 = 0$, $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 - 4x + 3 = 0$. Draw the figure.

19. Show that the radical center found in Exercise 18 is equidistant from the centers of the given circles.

20. If the centers of three non-concentric circles lie on the same straight line, where is the radical center?

Prove the following theorems analytically. Use general equations throughout.

21. The centers of all circles passing through the points of intersection of two given circles lie on the line of centers of the given circles.

22. The radical axis of two given circles is perpendicular to the line of centers of the given circles.

23. The radical axis of two given circles of equal radii is a perpendicular bisector of the segment joining the centers of the given circles.

24. The radical axes of three circles taken in pairs meet in a point.

THE PARABOLA

38. Definition and Construction. *A parabola is the locus of a point which moves so that its distance from a fixed point is always equal to its distance from a fixed straight line.*[1] The fixed point is called the **focus** and the fixed line the **directrix** of the parabola.

In Fig. 52, let F be the given point (focus) and DD' the given line (directrix). Draw a line through F perpendicular to DD' at C and let V be the mid-point of the segment CF. Since V is equidistant from C and F, it is, by definition, a point of the parabola. To construct other points, proceed as follows. Through any point Q, lying on the line through C and F and to the right of V, draw a line LL' parallel to the directrix DD'. Then with F as a center describe an arc of radius CQ, intersecting LL' in P and P'. Since $FP = CQ = MP$, the point P is equidistant

[1] The word *distance* as used in this definition refers to numerical distance; that is, distances are not directed. The same is true in the definitions of ellipse and hyperbola which are to follow.

from the focus and the directrix and, therefore, lies on the parabola. Likewise, P' is a point of the curve.

The line through C and F, which is seen to be a line of symmetry, is called the **axis** of the parabola. The point V, where the curve intersects its axis, is called the **vertex** of the parabola.

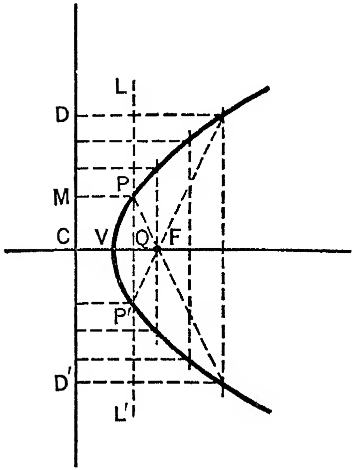
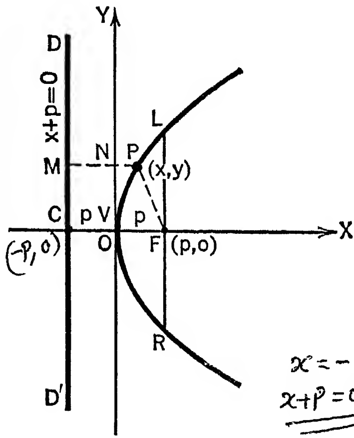


FIG. 52



$x = -p$

$x + p = 0$

FIG. 53

39. Equation of the Parabola. The simplest form of the equation of a parabola is obtained by using one of the coordinate axes as the axis of the parabola and taking the vertex at the origin. Thus, in Fig. 53, let F have coordinates $(p,0)$ and take V at O. Then the equation of the directrix is $x + p = 0$ since V is the mid-point of CF. For any point $P(x,y)$ on the curve, we have by definition $FP = MP$, or, by squaring both sides, $(FP)^2 = (MP)^2$. But

$$(FP)^2 = (x - p)^2 + y^2$$

by the distance formula, and

$$= (MN + NP)^2 = (p + x)^2.$$

Therefore $$(x - p)^2 + y^2 = (p + x)^2,$$

which reduces to $$y^2 = 4px. \tag{21}$$

This is the desired equation, for, as we have just shown, it is true for every point on the curve; and it is not true for any other point, since for a point not on the curve $FP \neq MP$, $(FP)^2 \neq (x - p)^2 + y^2$, $(x - p)^2 + y^2 \neq (p + x)^2$, and finally $y^2 \neq 4px$.

40. Discussion of the Equation. A glance at the equation $y^2 = 4px$ shows that it contains but two terms, the square of y and a constant times x. Hence it is satisfied by $x = 0$, $y = 0$ and remains unchanged when y is replaced by $-y$. This means that the locus of the equation passes through the origin and is symmetrical about the x-axis.

Reducing the equation to the form $y = \pm 2\sqrt{px}$, we see that p and x must be of like sign in order for y to be real and that for each value of x there are two values of y numerically equal but opposite in sign, these values of y increasing as x increases. Hence the curve opens to the right or the left according to whether p is positive or negative and extends indefinitely away from both coordinate axes.

When $x = p$, we find that $y = \pm 2p$. Therefore, the length of the chord through the focus perpendicular to the axis of the parabola is $4p$, the coefficient of x in the equation $y^2 = 4px$. This chord is called the **latus rectum** and is shown in Fig. 53 by the line LR.

If the focus is taken at the point $(0,p)$ on the y-axis and the equation of the parabola derived, we obtain

$$x^2 = 4py, \tag{22}$$

which represents a parabola with the origin as its vertex, the y-axis as its axis, the point $(0,p)$ as its focus and the line $y + p = 0$ as the equation of its directrix. It opens up or down according to whether p is positive or negative. The derivation of equation (22) is left as an exercise for the student.

Either of the above equations may be plotted by computing a table of values, but if just a sketch is desired this may be found by drawing the curve through the vertex and the ends of the latus rectum.

EXAMPLE. Discuss the equation $y^2 = -6x$ and sketch the curve.

The equation is satisfied by (0,0) and remains unchanged when $-y$ is substituted for y. Hence the curve passes through the origin and is symmetrical about the x-axis. By comparing the equation with the standard form $y^2 = 4px$, we see that $4p = -6$, or $p = -\frac{3}{2}$ and, therefore, the curve has its focus at $(-\frac{3}{2},0)$ and opens to the left. The equation of the directrix is $x - \frac{3}{2} = 0$, or $2x - 3 = 0$. When $x = -\frac{3}{2}$, $y = \pm 3$, and hence the coordinates of the ends of the latus rectum are $(-\frac{3}{2}, \pm 3)$. The length of the latus rectum is 6 units. With these facts known, the curve is readily drawn as in Fig. 54.

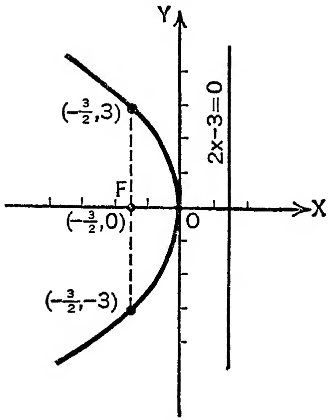


FIG. 54

EXERCISES

For each of the following parabolas, find the coordinates of the focus and ends of the latus rectum, and the equation of the directrix. Sketch each curve.

1. $y^2 = 8x$.
2. $x^2 = 8y$.
3. $y^2 = -4x$.
4. $2y^2 = -3x$.
5. $4x^2 = 5y$.
6. $x^2 + y = 0$.
7. $y^2 - 2x = 0$.
8. $3y^2 - 5x = 0$.
9. $6x^2 + 10y = 0$.
10. $2x^2 - 7y = 0$.

Find the equation of the parabola with vertex at the origin which passes through the point given below. Use the x-axis as the axis of the parabola.

11. $(3,1)$. **12.** $(-5,10)$. **13.** $(-4,-5)$.
14. $(3,-2)$. **15.** $(4,6)$. **16.** $(-8,4)$.
17. $(-2,-5)$. **18.** $(2,-3)$. **19.** $(-6,4)$.

Find the equation of the parabola with vertex at the origin which passes through the point given below. Use the y-axis as the axis of the parabola.

20. $(-5,6)$. **21.** $(-3,3)$. **22.** $(2,4)$.
23. $(-3,-5)$. **24.** $(6,-2)$. **25.** $(3,-5)$.
26. $(-6,-8)$. **27.** $(3,-4)$. **28.** $(-2,-5)$.

Find the equations of the following parabolas by means of the definition. Sketch each curve.

29. Focus at $(0,2)$, directrix $y + 2 = 0$.
30. Focus at $(0,-2)$, directrix $y - 2 = 0$.
31. Focus at $(0,0)$, directrix $4x - 6 = 0$.
32. Focus at $(2,-3)$, directrix $2y - 1 = 0$.
33. Focus at $(-4,-5)$, directrix $3x + 4 = 0$.
34. Focus at $(-2,4)$, directrix $3x - 5 = 0$.

Construct the following parabolas by the method of Art. 38.

35. Focus at $(6,0)$, directrix $x = 0$.
36. Focus at $(-2,3)$, directrix $y + 2 = 0$.
37. Focus at $(-5,-6)$, directrix $2x + 5 = 0$.
38. Focus at $(0,-5)$, directrix $2y - 7 = 0$.

41. Other Equations of the Parabola. We shall now make use of (21) found above to derive a more general equation of a parabola, namely, one whose vertex is not at the origin.

In Fig. 55, let the vertex of the desired parabola be at the point (h,k) and take the line $O'X'$ parallel to the x-axis as its axis. Erect a perpendicular $O'Y'$ to this axis at O'. Then the

equation of a parabola with its focus p units from the vertex and referred to these lines as coordinate axes is given by

$$y'^2 = 4px'.$$

In order to find its equation with reference to the original axes, we make use of the translation formulas of Art. 16, that is,

$$x' = x - h, \qquad y' \quad y - k.$$

Making these substitutions, we find that $y'^2 = 4px'$ becomes

$$(y - k)^2 = 4p(x - h), \quad (23)$$

which is the equation of a parabola with vertex at (h,k) and axis parallel to the x-axis; the equation of the axis being $y - k = 0$. The curve opens to the right or the left according to whether p is positive or negative.

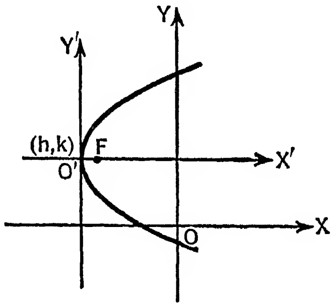


FIG. 55

In like manner, we find that

$$(x - h)^2 = 4p(y - k) \qquad (24)$$

is the equation of a parabola having the point (h,k) as its vertex and the line $x - h = 0$ as its axis. The sign of p again determines the direction in which the curve extends. The derivation of this equation is left as an exercise for the student.

It is readily seen that equations (23) and (24) may be written in the form

$$y^2 + Dx + Ey + F = 0 \quad \text{and} \quad x^2 + Dx + Ey + F = 0, \qquad (25)$$

respectively. Conversely, any equation of the form (25) may be reduced to one of the forms (23) or (24) by completing the square. Therefore, we say that *the equation $y^2 + Dx + Ey + F = 0$, where $D \neq 0$, represents a parabola with axis parallel to the*

axis, and $x^2 + Dx + Ey + F = 0$, *with* $E \neq 0$, *represents a parabola with axis parallel to the y-axis.* When $D = 0$, the first of equations (25) becomes $y^2 + Ey + F = 0$, which, by completing the square, may be written in the form $(2y + E)^2 = E^2 - 4F$. Hence, the locus consists of the line $2y + E = 0$ counted twice if $E^2 - 4F = 0$; of the pair of lines $2y = -E \pm \sqrt{E^2 - 4F}$ parallel to the x-axis if $E^2 - 4F > 0$; and is imaginary if $E^2 - 4F < 0$. A similar discussion may be made for the second of equations (25) when $E = 0$.

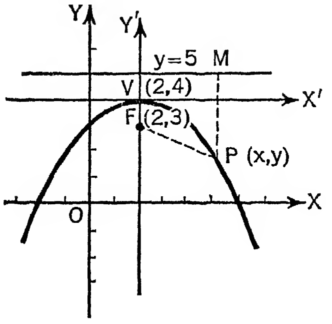


FIG. 56

EXAMPLE 1. Find the equation of the parabola having the point (2,3) as its focus and the line $y - 5 = 0$ as its directrix.

Using the definition of a parabola to obtain the equation, we have (Fig. 56)

$$FP = MP$$

or

$$\sqrt{(x-2)^2 + (y-3)^2} = y - 5.$$

We can reduce this relation to

$$x^2 - 4x + 4y - 12 = 0,$$

the desired equation, by squaring both sides and simplifying.

A second method of obtaining the equation is as follows. We know the vertex of a parabola lies on its axis midway between the directrix and focus, and, therefore, the coordinates of the vertex are (2,4). Since the focus of this particular parabola is below the directrix and at 2 units' distance, the curve extends downward and $p = -1$. Therefore, the equation of the curve referred to the lines through its vertex is $x'^2 = -4y'$. Since

$h = 2$ and $k = 4$, the coordinates of the vertex, this equation becomes

$$(x - 2)^2 = -4(y - 4), \qquad \text{or} \qquad x^2 - 4x + 4y - 12 = 0,$$

when referred to the original axes.

Example 2. Find the coordinates of the vertex and focus, the equations of the axis and directrix, and the length of the latus rectum of the parabola $2y^2 - 5x - 4y - 3 = 0$. Sketch the curve.

To reduce the equation to a standard form, we divide through by 2, transpose all terms except those containing y, and complete the square. We have

$$y^2 - 2y + 1 = \tfrac{5}{2}x + \tfrac{5}{2},$$

or

$$(y - 1)^2 = \tfrac{5}{2}(x + 1).$$

By comparing this equation with

$$(y - k)^2 = 4p(x - h),$$

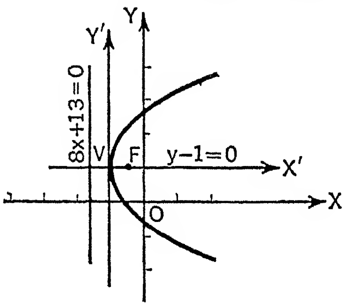


Fig. 57

we see that the parabola has its axis parallel to the x-axis, opens to the right, and has its vertex at $(-1,1)$. Also, since $4p = \frac{5}{2}$, we find that $p = \frac{5}{8}$, and hence the focus is at $(-1 + \frac{5}{8},1)$, or $(-\frac{3}{8},1)$. The equation of the axis is $y - 1 = 0$ and that of the directrix is $x = -1 - \frac{5}{8}$, or $8x + 13 = 0$. The length of the latus rectum is $\frac{5}{2}$ units; hence the coordinates of its end points are $(-\frac{3}{8}, 1 \pm \frac{5}{4})$, or $(-\frac{3}{8}, \frac{9}{4})$ and $(-\frac{3}{8}, -\frac{1}{4})$. The sketch is shown in Fig. 57.

EXERCISES

Find the equations of the following parabolas:

1. Focus at $(6,8)$ and equation of directrix $y - 2 = 0$.
2. Vertex at $(-2,3)$ and focus at $(1,3)$.

3. Focus at $(4,-2)$ and equation of directrix $x + 4 = 0$.
4. Vertex at $(2,3)$ and equation of directrix $x - 6 = 0$.
5. Focus at $(0,0)$ and vertex at $(-2,0)$.
6. Focus at $(0,-4)$ and equation of directrix $y = 0$.

Find the coordinates of the vertex, focus and ends of the latus rectum, and the equations of the axis and directrix of each of the following parabolas. Sketch the curves.

7. $y^2 + 8y + 8x = 0$.
8. $x^2 - 2x - 2y + 5 = 0$.
9. $y^2 + 12x - 6y - 39 = 0$.
10. $x^2 - 8x - 4y + 4 = 0$.
11. $4x^2 + 20x - y + 24 = 0$.
12. $2y^2 + 12y + 3x + 3 = 0$.
13. $3y^2 + 5y - 7x + 8 = 0$.
14. $2x^2 + 3x - 4y + 7 = 0$.

Find the equations of the following parabolas and sketch each curve:

15. Vertex at $(3,4)$, having axis parallel to the x-axis, and $p = 8$.
16. Vertex at $(3,-2)$, having axis parallel to the y-axis, and $p = -4$.
17. Vertex at $(1,2)$, having axis parallel to the x-axis, and passing through $(3,-1)$.
18. Vertex at $(-2,-3)$, having axis parallel to the y-axis, and passing through $(2,1)$.
19. Having axis parallel to the x-axis, vertices at $(-2,-1)$, and a latus rectum 5 units in length.
20. Passing through the points $(-2,3)$, $(\frac{1}{2},\frac{7}{4})$ and $(2,-5)$, with axis parallel to the x-axis.
21. Passing through the points $(1,-2)$, $(3,4)$ and $(-3,10)$, with axis parallel to the y-axis.
22. Find the equation of the parabola, the ordinate of every point of which is equal to its distance from the point $(0,4)$.
23. If a ball is thrown vertically upward with a velocity of 30 feet per second, the number of feet s above the ground is given approximately by the equation $s = 30t - 16t^2$. Write this equation in a standard form of the parabola, where now the coordinates of a point on the curve are (t,s), and sketch the curve. When will the ball reach the ground? Does the path of the ball coincide with the parabola?
24. The distance between the ends of a parabolic arch is 10 feet and the center of the arch is 3 feet above a horizontal line joining the end

points. Find the equation of the parabola if the horizontal line mentioned above and a vertical line passing through the center of the arch and meeting the horizontal line at right angles are taken as the coordinate axes. What are the coordinates of the focus?

25. Find the equation of the circle which passes through the vertex and the ends of the latus rectum of the parabola $x^2 = 12y$.

THE ELLIPSE

42. Definition and Equation. *An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is a constant.* The two fixed points are called **foci** and the midpoint of the line segment joining them is called the **center** of the ellipse.

To obtain a simple form of the equation of the ellipse, let us take the foci on the x-axis and the center at the origin (Fig. 58). Then, if the distance between the foci F' and F is assumed to be $2c$ units in length, the coordinates of these points are $(-c,0)$ and $(c,0)$, respectively. Furthermore, if the sum of the distances of any point $P(x,y)$ on the ellipse from the foci is denoted by $2a$, we have by definition

$$F'P + FP = 2a.$$

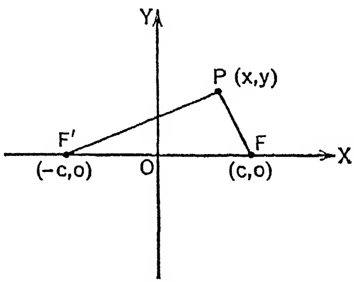


FIG. 58

By looking at the triangle $F'PF$ in Fig. 58, we see at once that $2a > 2c$ for any point P not on the segment $F'F$, since the sum of two sides of a triangle is always greater than the third side. This being true, we shall take $a > c$ in the following discussion.

Expressing the above relation in terms of coordinates, we have

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a. \tag{26}$$

which, by transposing the second radical,[1] squaring and reducing, becomes

$$a^2 - cx = a\sqrt{(x - c)^2 + y^2}.$$

Squaring again and simplifying, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

But $a^2 - c^2$ is a positive number since $a > c$; hence $a^2 > c^2$. Let us call this number b^2, where b is real, and make the substitution $b^2 = a^2 - c^2$. Our equation then takes the form

$$b^2x^2 + a^2y^2 = a^2b^2, \tag{27}$$

or, by dividing both sides by a^2b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{28}$$

What we have done so far is to show that every point which satisfies the condition $F'P + FP = 2a$ has coordinates which satisfy equation (28). We must now show the converse, namely, that every point whose coordinates satisfy equation (28) must also satisfy (26); hence be a point on the ellipse. If we assume that the coordinates of a point satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we obtain, by reversing the steps in the above derivation, the four equations,

$$\pm\sqrt{(x + c)^2 + y^2} \quad \pm\sqrt{(x - c)^2 + y^2} = 2a.$$

It will now be necessary to show that the only equation of these four which may be interpreted geometrically is the one that leads to the relation $F'P + FP = 2a$. To do this, let us examine the several equations. If a plus sign is used with one radical and a minus sign with the other, we obtain the relations $F'P - FP = 2a$

[1] The student should satisfy himself that the same final result may be obtained by transposing the first radical.

and $-F'P + FP = 2a$. Now since $2a > 2c$, the interpretation of these expressions is that the difference between two sides of the triangle $F'PF$ is greater than the third side. But this is known to be false, since the difference between two sides of a triangle is always less than the third side. Using both minus signs in the left member, we have $-F'P - FP = 2a$, which is likewise false because $2a$ is a positive number and cannot be equal to the negative quantity $-F'P - FP$. Hence there remains the relation $F'P + FP = 2a$, which we know to be the condition that the point P lies on the ellipse.

Thus we have shown that the equation of the ellipse with center at the origin, foci at $(\pm c, 0)$, *and having* $2a$ *as a constant, is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

43. Discussion of the Equation. The ellipse which is represented algebraically by equation (28) is symmetrical about both coordinate axes and the origin. The truth of this statement may be verified by observing that the equation remains unaltered when x is replaced by $-x$; when y is replaced by $-y$; and finally, when both x and y are replaced by $-x$ and $-y$, respectively.

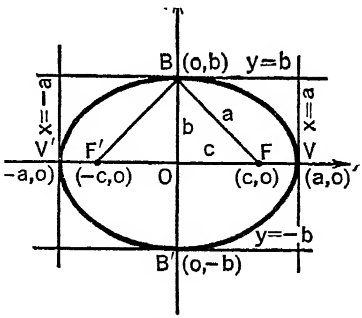


FIG. 59

Solving the equation of the ellipse for y in terms of x and for x in terms of y, we have

and $x =$

The first of these equations shows that the only values of x which give real values of y are those for which $x^2 \leqq a^2$. Like-

wise, from the second equation, values of y such that $y^2 \leqq b^2$ are the only ones which give real values of x. Hence (Fig. 59) the curve lies between the lines $x = \pm a$ and $y = \pm b$. If $x = \pm a$, we find that $y = 0$, and if $y = \pm b$, $x = 0$; hence the curve cuts the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$.

The line segment $V'V$, of length $2a$, passing through the foci is called the **major axis,** while the chord $B'B$, of length $2b$, passing through the center perpendicular to the major axis is called the **minor axis.** The lengths a and b are called the **semi-major** and **semi-minor** axes, respectively. The end points, V' and V, of the major axis are known as the **vertices** of the ellipse.

As we have already seen, the relationship between the constants a, b and c is expressed by the equation $a^2 = b^2 + c^2$. Interpreted geometrically, this means that a line drawn from a focus to an end of the minor axis has the same length as the semi-major axis.

The chord through either focus perpendicular to the major axis is called the **latus rectum.** Its length is found by substituting $x = c$ or $x = -c$ in the equation of the ellipse and solving for y. This gives

$$y = \pm\frac{b}{a}\sqrt{a^2 - c^2} = \pm\frac{b^2}{a}, \text{ since } a^2 - c^2 = b^2.$$

Hence the length of the latus rectum is $\frac{2b^2}{a}$, since it is the double ordinate at a focus.

The value of the ratio $\frac{c}{a}$ indicates the shape of the ellipse, since for a of fixed length, the curve flattens out as $c \to a$ and approaches a circle of radius a as $c \to 0$. The ratio takes values between 0 and 1 since $c < a$. It is called the **eccentricity** and is designated by the letter e, that is, $e = \frac{c}{a}$.

The equation of the ellipse with major axis along the y-axis and

foci at $(0, \pm c)$ *is given by*

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

Since this equation may be obtained from (28) by interchanging the variables x and y, it is seen that the major and minor axes, the latus rectum and the eccentricity have the same values here as in the case discussed above.

44. Construction of an Ellipse. An ellipse may be constructed by means of points in the following manner. Lay off the major axis $V'V$ as in Fig. 60 and locate the foci F' and F. Let M be any point on the line segment $F'F$. Then with the foci as centers and a radius MV, draw arcs above and below the major axis. With the same centers and a radius MV', draw arcs intercepting those just found. This will give four points of the ellipse, and others may be found by varying M. The validity of this construction is seen by calling one of the points P_1 and observing that $MV = F'P_1$ and $MV' = FP_1$, and therefore

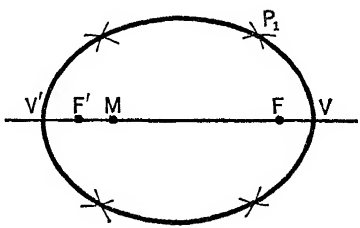


FIG. 60

$$F'P_1 + FP_1 = MV + MV' = V'V,$$

the length of the major axis.

For just a sketch of the ellipse, draw the curve through the x and y intercepts and the extremities of the latera recta.

EXAMPLE. Determine the semi-major and the semi-minor axes, the coordinates of the foci and vertices, the length of the latus rectum, the eccentricity, and sketch the ellipse

$$9x^2 + 4y^2 = 36.$$

To reduce the equation to standard form, let us divide both sides by 36 and obtain

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Since the larger of the two numbers 9 and 4 is in the term containing y^2, we know that the major axis lies along the y-axis. The semi-major axis is $a = 3$ and the semi-minor axis is $b = 2$. The coordinates of the vertices are $(0, \pm 3)$ and those of the ends of the minor axis are $(\pm 2, 0)$. From the relation $c^2 = a^2 - b^2$, we find $c = \sqrt{5}$; hence the coordinates of the foci are $(0, \pm\sqrt{5})$. The length of the latus rectum is $\frac{2b^2}{a} = \frac{8}{3}$, and the coordinates of the extremities are $(\pm\frac{4}{3}, \pm\sqrt{5})$.

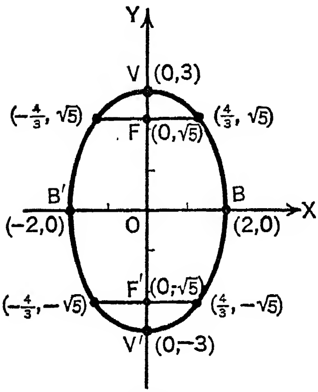


FIG. 61

The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{5}}{3}$. Fig. 61 shows the sketch.

EXERCISES

Find the semi-axis, the foci, the vertices, the latus rectum and the eccentricity of the following ellipses. Sketch the curve.

1. $x^2 + 2y$ **2.** $9x^2 + y^2 = 36.$
3. $3x^2 + y^2 = 12.$ **4.** $7x^2 + 16y^2 = 112.$
5. $8x^2 + 4y^2 = 32.$ **6.** $4x^2 + y^2 = 4.$
7. $16x^2 + 36y^2 = 576.$ **8.** $25x^2 + 9y^2 = 225.$
9. $9x^2 + 36y^2 = 324.$ **10.** $3x^2 + 2y^2 = 1.$

Write the equations of the following ellipses and sketch the curves.

11. Major axis 10 and foci at $(0, \pm 3)$.

12. Minor axis 2 and vertices at $(\pm 4, 0)$.

13. Center at the origin, major axis 6, latum rectum $\frac{8}{3}$ and foci on y-axis.

14. Eccentricity $\frac{1}{2}$, distance between foci 5, center at the origin and major axis along x-axis.

15. Center at the origin, minor axis 8, eccentricity $\frac{2}{3}$ and foci on the

16. Vertices $(\pm 4,0)$ and passing through $(3,2)$.

17. Vertices $(\pm 8,0)$ and eccentricity $\frac{3}{4}$.

18. Center at the origin, major axis along x-axis and passing through $(-5,0)$ and $(3,\frac{16}{5})$.

19. Center at the origin, minor axis along y-axis and passing through $(4,-2)$ and $(-2,-3)$.

20. Center at the origin, major axis twice the minor axis and passing through $(4,2)$.

21. Find the locus of a point which moves so that the sum of its distances from $(0,\pm 3)$ is 12.

22. Find the locus of a point which moves so that the sum of its distances from $(\pm 2,0)$ is 8.

23. Find the eccentricity of an ellipse when the latus rectum is three-fourths the minor axis.

24. Find the area of the square inscribed in the ellipse

$$b^2x^2 + \quad = a^2b^2.$$

25. Find the equation of the circle passing through the extremities of the latera recta of the ellipse $a^2x^2 + b^2y^2 = a^2b^2$.

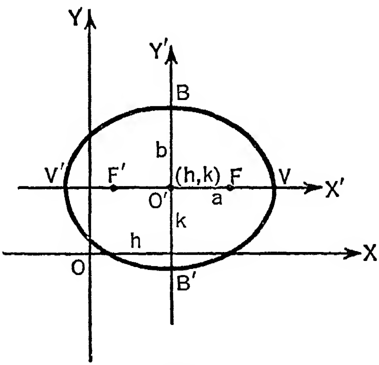


FIG. 62

45. Other Equations of the Ellipse. A more general equation of the ellipse may be found in the following manner. In Fig. 62, let (h,k) be the center of the ellipse whose equation we wish to find and let the lines $O'X'$ and $O'Y'$, drawn parallel to the coordinate axes, be its axes. Then, if the major axis is of length

$2a$ and coincides with $O'X'$ and if the foci are on this line c units to the right and left of O', we know that the equation of the curve referred to its axes is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

By making use of the formulas

$$x' = x - h \qquad \text{and} \qquad y' = y - k$$

to translate the center (h,k) to the point O, we have

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \tag{30}$$

which is the desired equation referred to the x and y axes.

Similarly, if the major axis is taken parallel to the y-axis, our equation becomes

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1.$$

By expanding the binomial terms and clearing of fractions, either of the above equations may be written

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{32}$$

It will now be shown, conversely, that every equation of the form (32), in which A and C have the same sign and are not zero, represents an ellipse with axes parallel to the coordinate axes.

By the process of completing squares, equation (32) may be written

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = \frac{D^2}{4A} + \frac{E^2}{4C} - F,$$

or
$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F.$$

If we let $h = -\dfrac{D}{2A}$, $k = -\dfrac{E}{2C}$ and $H = \dfrac{D^2}{4A} + \dfrac{E^2}{4C} - F$, this equation becomes

$$A(x - h)^2 + C(y - k)^2 = H,$$

or

$$\frac{(x - h)^2}{\frac{H}{A}} + \frac{(y - k)^2}{\frac{H}{C}} = 1,$$

which is of the form (30) or (31) according as $\dfrac{H}{A}$ is greater than or less than $\dfrac{H}{C}$. It therefore represents an ellipse with axes parallel to the coordinate axes. For the ellipse to be real H must have the same sign as A and C; if $H = 0$ the locus is a *point ellipse.*

Example 1. The foci of an ellipse are at (2,4) and (2,−6), and the semi-major axis is $a = 6$. Determine the elements b and c, find the equation and sketch the curve.

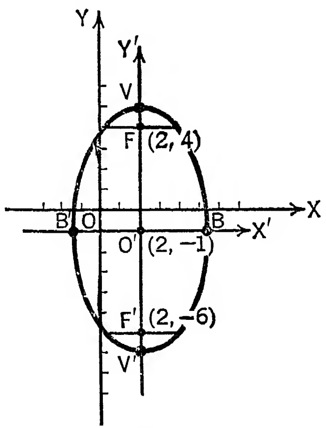


Fig. 63

Since the foci are on the line $x = 2$, shown in Fig. 63, we know that the major axis is parallel to the y-axis. The center of the curve is at $(2, -1)$, the mid-point of the segment joining (2,4) and (2,−6). The distance between the foci is $2c = 10$. So, $c = 5$, and since $a = 6$, we have $b^2 = a^2 - c^2 = 11$. Therefore, the equation of the ellipse referred to the x' and y' axes is

$$\frac{y'^2}{36} + \frac{x'^2}{11} = 1.$$

When referred to the x and y axes, it becomes

$$\frac{(y+1)^2}{36} + \frac{(x-2)^2}{11} = 1,$$

or
$$36x^2 + 11y^2 - 144x + 22y - 241 = 0.$$

The sketch is drawn by using the known elements $a = 6$, $b = \sqrt{11}$, $c = 5$ and the length of the latus rectum $\frac{2b^2}{a} = \frac{11}{3}$.

EXAMPLE 2. Find the center, semi-axes, vertices, foci, latus rectum and eccentricity of the ellipse whose equation is

$$49x^2 + 144y^2 - 196x - 720y - 668 = 0.$$

By completing the squares, the equation may be written

$$49(x^2 - 4x + 4) + 144(y^2 - 5y + \tfrac{25}{4}) = 1764,$$

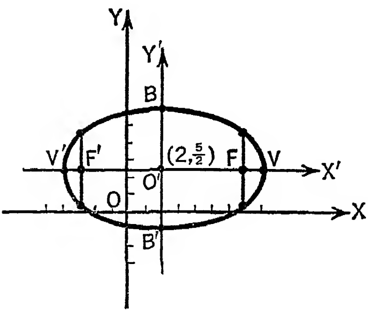


FIG. 64

which becomes, when reduced to standard form,

$$\frac{(x-2)^2}{36} + \frac{(y-\frac{5}{2})^2}{\frac{49}{4}} = 1.$$

This final equation shows that the center of the curve is at the point $(2,\frac{5}{2})$ and that the major axis coincides with the line $2y - 5 = 0$ and the minor axis with the line $x - 2 = 0$. The semi-axes are $a = 6$, $b = \frac{7}{2}$, and the coordinates of the vertices are $(2 \pm 6,\frac{5}{2})$, or $(8,\frac{5}{2})$ and $(-4,\frac{5}{2})$. Since $c^2 = a^2 - b^2 = \frac{95}{4}$, we have $c = \frac{1}{2}\sqrt{95}$; hence the foci are at $(2 \pm \frac{1}{2}\sqrt{95},\frac{5}{2})$. The length of the latus rectum is $\frac{2b^2}{a} = \frac{49}{12}$, while the eccentricity is $e = \frac{c}{a} = \frac{1}{12}\sqrt{95}$. Figure 64 shows the sketch.

EXERCISES

Find the center, semi-axes, vertices, foci, latus rectum and eccentricity of the following ellipses. Sketch each curve.

1. $x^2 + 9y^2 - 4x - 5 = 0$.
2. $16x^2 + 7y^2 - 64x + 14y - 377 = 0$.
3. $25x^2 + 9y^2 - 10x - 12y - 220 = 0$.
4. $5x^2 + 9y^2 - 10x + 24y - 24 = 0$.
5. $3x^2 + 4y^2 - 12x + 24y = 0$.
6. $5x^2 + y^2 - 10x + 8y - 4 = 0$.
7. $4x^2 + 25y^2 - 8x + 50y - 171 = 0$.
8. $x^2 + 4y^2 - 8x - 16y + 16 = 0$.
9. $x^2 + 2y^2 + x - 4y - 20 = 0$.

In each of the following cases find the equation of the ellipse having its axes parallel to the coordinate axes and satisfying the given conditions. Sketch the curve.

10. Center at $(-1,-2)$, $a = 4$, $b = 2$, and major axis parallel to the x-axis.

11. Vertices at $(-3,6)$ and $(-3,-2)$, and $e = \frac{2}{3}$.

12. Center at $(3,4)$, $e = \frac{1}{2}$, $a = 6$, and major axis parallel to the x-axis.

13. Foci at $(6,3)$ and $(6,-9)$, and $a = 10$.

14. Vertices at $(0,0)$ and $(-5,0)$, and one focus at $(-2,0)$.

15. Center at $(1,2)$, major axis parallel to the x-axis, and passing through $(1,1)$ and $(3,2)$.

16. Major axis parallel to the y-axis, center at $(-3,4)$, $e = \frac{4}{5}$, and passing through $(6,7)$.

17. Vertices at $(1,-4)$ and $(1,6)$, and having one focus on the line $2y + x - 7 = 0$.

18. Major axis parallel to the y-axis, center at the intersection of the lines $x - y - 2 = 0$ and $2x + y - 4 = 0$, latus rectum of length 6, and $a = 4$.

19. Foci at the intersections of the circle $x^2 + y^2 - 4y - 5 = 0$ and the parabola $2x^2 - 9y = 0$, and $e = \frac{3}{4}$.

20. $Ax^2 + Cy^2 + Dx + Ey + F = 0$ divided by A takes the form $x^2 + ay^2 + bx + cy + d = 0$, which contains four arbitrary constants.

Find the equation of the ellipse with axes parallel to the coordinate axes and passing through the points (0,1), (1,2), (2,1) and (2,0).

21. A line segment l of constant length moves in such a way that its ends are on the coordinate axes. A point P is on l at a units from its intersection with the x-axis and b units from its intersection with the y-axis. Find the locus of P.

THE HYPERBOLA

46. Definition and Equation. *A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is a constant.* The two fixed points are called the **foci**, and the mid-point of the line segment joining them is called the **center** of the hyperbola.

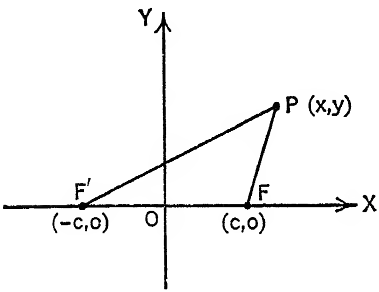


FIG. 65

A simple form of the equation of the curve may be obtained by taking the foci on the x-axis and the center at the origin. Thus (Fig. 65), if the coordinates of the foci F' and F are $(-c,0)$ and $(c,0)$, respectively, and if $P(x,y)$ is any point on the hyperbola such that the difference of its distances from the foci is $2a$, we have by definition [1]

$$F'P - FP = 2a.$$

In terms of coordinates, this condition becomes

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a,$$

which may be reduced to the form

$$(c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$$

[1] The condition of the definition may also be written $FP - F'P = 2a$.

by following the steps used in finding the equation of the ellipse (Art. 42).

Since the difference of two sides of a triangle is always less than the third side, we have for triangle $F'PF$ of the figure, $F'P - FP < F'F$, or $2a < 2c$. Hence $a < c$ and $c^2 - a^2$ is a positive number. Let b^2 represent this number, where b is real, and make the substitution $b^2 = c^2 - a^2$. Our equation then has the form

$$b^2x^2 - a^2y^2 = a^2b^2,$$

or, by dividing through by a^2b^2,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{33}$$

We have shown, therefore, that every point which is on the hyperbola has coordinates that satisfy equation (33). The converse is likewise true, namely, that every point whose coordinates satisfy (33) is a point which lies on the hyperbola. This part of the proof will be left as an exercise for the student.

47. Discussion of the Equation. As in the corresponding case of the ellipse, the hyperbola which is represented algebraically by equation (33) is symmetrical about both axes and the origin.

Solving equation (33) first for y and then for x, we have

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2} \qquad \text{and} \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

We see from the first of these equations that in order for y to be real, x must take values such that $x^2 \geqq a^2$; that is, no values of x between $x = -a$ and $x = a$ will give a point on the curve. The second equation shows that x is real for all real values of y. If $y = 0$, we find that $x = \pm a$, and if $x = 0$, y is imaginary. Hence the curve (Fig. 66) cuts the x-axis at $(\pm a,0)$ but does not intersect the y-axis. It consists of two branches lying outside of the lines $x = \pm a$ and extending indefinitely away from both

coordinate axes. The line through the foci is called the *principal axis* and the segment $V'V$, of length $2a$, is called the **transverse axis.** The line segment on the y-axis between the points $(0,b)$

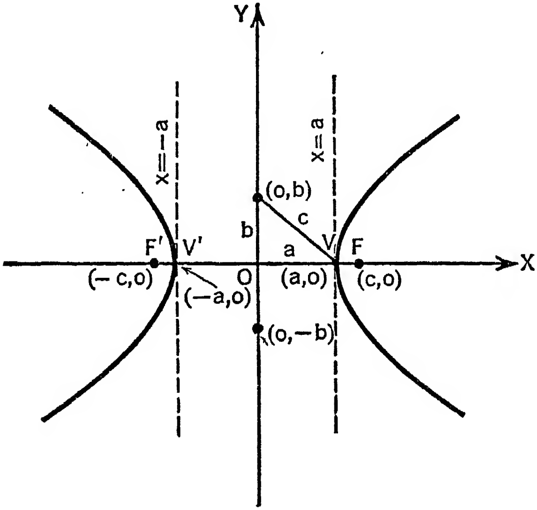


FIG. 66

and $(0,-b)$, of length $2b$, is called the **conjugate axis.** The lengths a and b are called the **semi-transverse axis** and **semi-conjugate axis,** respectively. The points V' and V, at the ends of the transverse axis, are called the **vertices** of the hyperbola.

From the relationship $c^2 = a^2 + b^2$, we see that the distance from the center to a focus is the same as the distance from a vertex to an end of the conjugate axis.

The chord through a focus perpendicular to the principal axis is called the **latus rectum.** Its length, $\frac{2b^2}{a}$, is found in the same way as for the ellipse.

The eccentricity, $e = \frac{c}{a}$, is seen to be greater than 1 for the

hyperbola, since $c > a$. Its value indicates the shape of the curve.

The equation of the hyperbola with transverse axis along the y-axis and foci at $(0, \pm c)$ *is given by*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{34}$$

The transverse and conjugate axes, the latus rectum and the eccentricity have the same values here as in the case discussed above.

48. Construction of a Hyperbola. A point by point construction of a hyperbola is similar to that of an ellipse. Locate the foci, F' and F, and the vertices, V' and V, as in Fig. 67. Let M be a point on the principal axis to the right of F. Then, with the foci as centers and a radius MV, describe arcs above and below the principal axis. With the same centers and a radius MV', describe arcs intersecting those just drawn. The four points thus found lie on the hyperbola. Other points may be constructed by varying M where M may coincide with F' or F, or may fall to the left of F' as well as to the right of F. The construction is seen to be correct from the following argument, where P_1 is taken as a typical point located as described above.

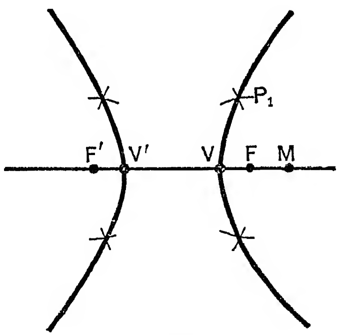


FIG. 67

$$MV' = F'P_1 \qquad \text{and} \qquad MV = FP_1.$$

Therefore

$$F'P_1 - FP_1 = MV' - MV = VV',$$

the length of the transverse axis.

49. Asymptotes of a Hyperbola. We have seen from the discussion of Art. 47 that the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ consists of two branches opening outward to the right and left of the y-axis. Now let us draw a line through the origin (Fig. 68) intersecting these branches in the points P and P'. If $y = mx$ is the equation of this line, we obtain, by solving it simultaneously with the equation of the hyperbola,

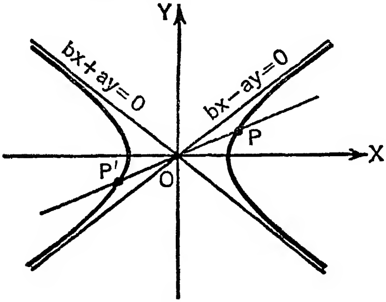


FIG. 68

$$x = \pm \frac{ab}{\sqrt{b^2 - }}$$

as the abscissas of the points of intersection of the line and the curve. As P moves to the right [1] along the curve, or P' to the left, the numerical value of x increases without limit, and, therefore, since a and b are fixed numbers, the denominator of the above fraction approaches zero. When $b^2 - a^2m^2 = 0$ we have $m = \pm\frac{b}{a}$, and the equation $y = mx$ becomes

$$y = \pm\frac{b}{a}x, \qquad \text{or} \qquad bx \pm ay = 0.$$

Hence, there are two lines, $bx - ay = 0$ and $bx + ay = 0$, passing through the origin with slopes $\frac{b}{a}$ and $-\frac{b}{a}$, respectively, to which the hyperbola comes nearer and nearer as the numerical value of x increases.

If we define *an* ***asymptote*** *of a curve as a straight line such that the perpendicular distance from the line to a point on the curve*

[1] It should be observed that P may move along the lower half of the right-hand branch as well as along the upper.

becomes and remains less than any assignable quantity, as the point on the curve recedes indefinitely from the origin, then, as will be shown in Exercise 31, page 103, the lines $bx \pm ay = 0$ are *asymptotes* of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

An easy way to find the equations of the asymptotes is to make the right member of the equation of the hyperbola zero and factor. Thus

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \qquad \text{or} \qquad b^2x^2 - a^2y^2 = 0$$

factors into $bx - ay = 0$ and $bx + ay = 0$, the equations of the asymptotes.

If the equation of the hyperbola is $b^2y^2 - a^2x^2 = a^2b^2$, showing that the foci are on the y-axis, the equations of the asymptotes are $by \pm ax = 0$.

The asymptotes are very useful in sketching a hyperbola. Let us take the transverse axis $2a$ and the conjugate axis $2b$ as shown in Fig. 69 and construct a rectangle with its center at the center of the hyperbola and sides $2a$ and $2b$ parallel to the transverse and conjugate axes, respectively. Since the diagonals of this rectangle have slopes $\frac{b}{a}$ and $-\frac{b}{a}$, they become, when produced, the asymptotes of the hyperbola. Thus to sketch a hyperbola, we draw the asymptotes first and use them as guide lines for the curve which is drawn tangent to the rectangle at a vertex and passing through the extremities of the latera recta. It is to be observed that a circle having the diagonals as diameters will pass through the foci of the hyperbola.

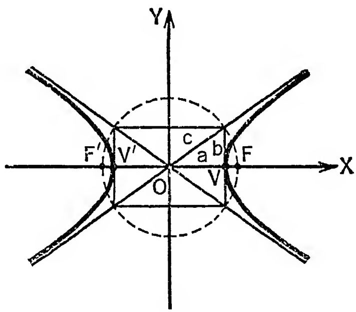


Fig. 69

EXAMPLE. Find the values of a, b, c, the coordinates of the foci, vertices and ends of the latera recta, the length of a latus rectum and the equations of the asymptotes for the hyperbola $49y^2 - 16x^2 = 196$. Sketch the curve.

Reducing the equation to standard form, we have

$$\frac{y^2}{4} - \frac{x^2}{\frac{49}{4}} = 1.$$

Hence $a = 2$, $b = \frac{7}{2}$, $c = \sqrt{a^2 + b^2} = \frac{1}{2}\sqrt{65}$ and $e = \frac{c}{a} = \frac{1}{4}\sqrt{65}$. Since the term containing y is positive, we know that the transverse axis is along the y-axis. Hence the coordinates of the desired points are: foci, $(0, \pm\frac{1}{2}\sqrt{65})$; vertices, $(0, \pm 2)$; ends of the latera recta, $(\pm\frac{49}{8}, \pm\frac{1}{2}\sqrt{65})$. The length of a latus rectum is $\frac{2b^2}{a} = \frac{49}{4}$. The equations of the asymptotes may be found by factoring $49y^2 - 16x^2 = 0$; they are $7y - 4x = 0$ and $7y + 4x = 0$. Figure 70 shows the sketch.

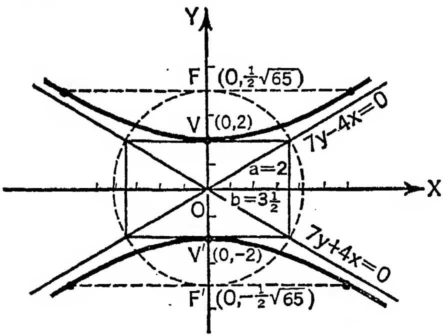


FIG. 70

EXERCISES

Find the values of a, b, c and e, the coordinates of the foci, vertices and ends of the latera recta, the length of a latus rectum and the equations of the asymptotes for the following hyperbolas. Sketch the curves.

1. $9x^2 - 16y^2 = 144$.
2. $8y^2 - x^2 = 8$.
3. $81y^2 - 144x^2 = 11664$.
4. $x^2 - 9y^2 = 9$.
5. $y^2 - x^2 = 16$.
6. $16x^2 - 20y^2 = 320$.
7. $4x^2 - 25y^2 = 100$.
8. $x^2 - 4y^2 = 4$.
9. $x^2 - y^2 = 64$.
10. $9y^2 - 4x^2 = 25$.

Construct the following hyperbolas by the method of Art. 48.

11. Foci $(\pm 3,0)$, $a = 2$.

12. Foci $(0,\pm 5)$, $b = 3$.

13. $2a = 4$, $2c = 7$.

14. Semi-conjugate axis 6, $e = \frac{5}{4}$.

15. $4x^2 - 9y^2 = 16$.

16. $x^2 - y^2 = 25$.

Write the equations of the hyperbolas with centers at the origin, axes along the coordinate axes and satisfying the following conditions. Sketch each curve.

17. One vertex at $(0,8)$ and $e = 2$.

18. Transverse axis 6 and foci at $(\pm 12,0)$.

19. Vertices at $(\pm 4,0)$ and latus rectum 18.

20. Conjugate axis 6 and foci at $(\pm\sqrt{13},0)$.

21. Eccentricity $\frac{5}{3}$, transverse axis 12 and foci on the x-axis.

22. Foci at $(0,\pm 3)$ and latus rectum 5.

23. Foci on the x-axis and passing through $(-3,1)$ and $(9,5)$.

24. Vertices on the y-axis and passing through $(0,3)$ and $(12,5)$.

25. Passing through the point $(\frac{16}{3},5)$ and having asymptotes $3x \pm 2y = 0$.

26. Passing through the point $(5,8)$ and having asymptotes $2x \pm y = 0$.

27. Vertices on the x-axis midway between the center and foci, and latus rectum 16.

28. Transverse axis on the y-axis, passing through $(3,4)$, and $e = \frac{3}{2}$.

29. Conjugate axis on the y-axis, and passing through $(-4,2)$ and $(6,-8)$.

30. Having the line $5x + 4y = 0$ as an asymptote and passing through $(-6,4)$.

31. Show that the lines $bx \pm ay = 0$ are asymptotes of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$. Hint: Consider a point (x_1,y_1) in the first quadrant and lying on the curve. Then, the perpendicular distance from the line $bx - ay = 0$ to this point is $d = -\frac{\quad - ay_1}{\sqrt{a^2}}$ by (16). But, since (x_1,y_1) satisfies the equation of the hyperbola, we may write

$$d = -\frac{a^2b}{\sqrt{a^2 + b^2}(x_1 + \sqrt{x_1^2 - a^2})}$$

and observe that d approaches zero as x_1 increases without limit. Similarly, $bx + ay = 0$ may be shown to be an asymptote of the curve.

50. Conjugate Hyperbolas. *Two hyperbolas are said to be* ***conjugate*** *when the transverse axis of each is the conjugate axis of the other.* Thus the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

represent conjugate hyperbolas. It is seen at once that the foci

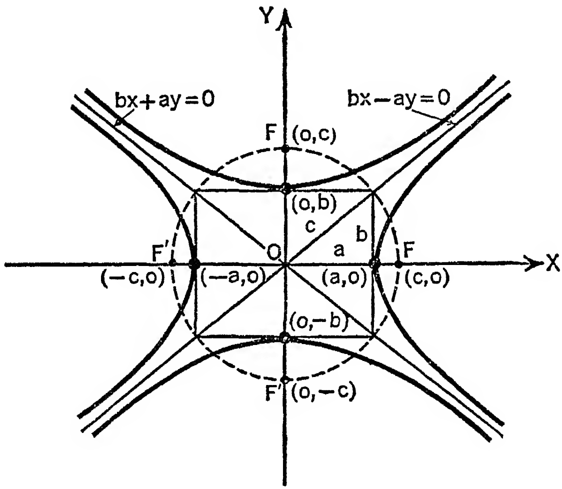


FIG. 71

are equidistant from the center, because $a^2 + b^2$ has the same value in both cases. Also, the two hyperbolas have common asymptotes, for the left member of each equation, when set equal to zero, factors into $bx \pm ay = 0$. These facts are shown graphically in Fig. 71.

Since $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is the same as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, we are able

to write the equation of the hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by changing the sign of the constant term.

51. Equilateral Hyperbola. From our study of the ellipse, we know that the locus becomes a circle when the major and minor axes are equal. In the corresponding case of a hyperbola, when the transverse and conjugate axes are of the same length, the hyperbola is said to be **equilateral.** Thus $b^2x^2 - a^2y^2 = a^2b^2$ becomes $x^2 - y^2 = a^2$ when $b = a$, and the second equation represents an equilateral hyperbola with center at the origin and foci on the x-axis.

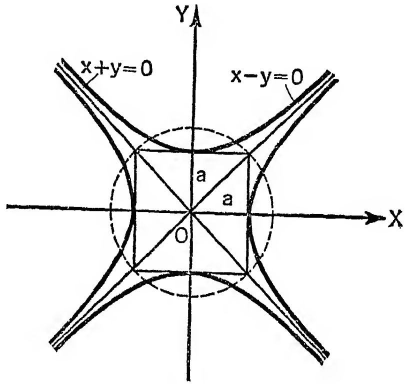


FIG. 72

Since the asymptotes of an equilateral hyperbola meet at right angles—for example, the asymptotes of $x^2 - y^2 = a^2$ are $x \pm y = 0$, two perpendicular lines—such a hyperbola is often called **rectangular.** It is to be observed (Fig. 72) that the rectangle associated with a hyperbola is now a square.

The hyperbola conjugate to an equilateral hyperbola is the same curve, although it is differently placed. Thus, the conjugate of $x^2 - y^2 = a^2$ is $x^2 - y^2 = -a^2$, or $y^2 - x^2 = a^2$, which represents the original curve with foci on the y-axis.

52. Other Equations of the Hyperbola. By making use of the translation formulas, as in the case of the ellipse (Art. 45), *the equation of a hyperbola having its center at (h,k) and transverse axis parallel to the x-axis is found to be*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{35}$$

If the transverse axis is parallel to the y-axis, the corresponding equation is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \tag{36}$$

Either of these equations may be expressed in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{37}$$

by expanding the binomial terms and clearing of fractions. Conversely, every equation of the form (37), in which A and C have different signs, represents a hyperbola with axes parallel to the coordinate axes. The argument is similar to that used for the ellipse and will not be given here.

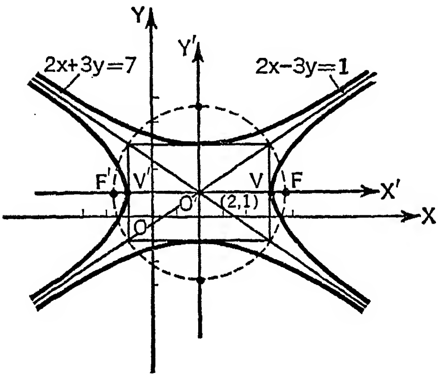


FIG. 73

EXAMPLE. Determine the values of a, b, c and e; the coordinates of the center, foci and vertices; the lengths of the latus rectum, transverse and conjugate axes; and the equations of the principal axis and asymptotes for the hyperbola

$$4x^2 - 9y^2 - 16x + 18y - 29 = 0.$$

Find the equation of the conjugate hyperbola and sketch both curves.

Completing the squares, we may express the equation in the form

$$4(x-2)^2 - 9(y-1)^2 = 36,$$

or

$$\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 1.$$

The term containing x being positive, we know that the transverse axis is parallel to the x-axis and the equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

We find that $a = 3$ and $b = 2$; hence $c = \sqrt{a^2 + b^2} = \sqrt{13}$, and $e = \frac{c}{a} = \frac{1}{3}\sqrt{13}$. The coordinates of the center are $(2,1)$ and therefore the foci are at the points $(2 \pm \sqrt{13},1)$, while the vertices are at $(5,1)$ and $(-1,1)$. The lengths of the latus rectum, transverse and conjugate axes are respectively $\frac{2b^2}{a}$ $\frac{8}{3}$, $2a = 6$ and $2b = 4$. The equation of the principal axis is $y - 1 = 0$, and the asymptotes are represented by $2x - 3y - 1 = 0$ and $2x + 3y - 7 = 0$. These last equations are found by factoring $4(x-2)^2 - 9(y-1)^2 = 0$. The equation of the conjugate hyperbola is

$$\frac{(y-1)^2}{4} - \frac{(x-2)^2}{9} = 1,$$

or

$$4x^2 - 9y^2 - 16x + 18y + 43 = 0.$$

The sketch is shown in Fig. 73.

EXERCISES

Find the coordinates of the center, foci and vertices, the length of a latus rectum, the equations of the asymptotes and conjugate hyperbola for each of the following. Sketch the hyperbola and its conjugate on the same set of axes.

1. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1.$
2. $\frac{(x+3)^2}{64} - \frac{(y-4)^2}{225} = 1.$
3. $\frac{(y+3)^2}{4} - \frac{(x+4)^2}{25} = 1.$
4. $\frac{(y-5)^2}{36} - \frac{(x-2)^2}{36} = 1.$
5. $9y^2 - 4x^2 - 8x + 18y + 41 = 0.$

6. $6x^2 - 10y^2 + 12x - 80y - 214 = 0$.

7. $9x^2 - y^2 + 36x + 6y + 18 = 0$.

8. $9y^2 - 4x^2 + 4x + 12y - 39 = 0$.

9. $2y^2 - 3x^2 + 12x + 12y = 0$.

10. $4x^2 - 5y^2 - 16x - 20y - 24 = 0$.

11. $x^2 - y^2 + 4x - 6y - 9 = 0$. **12.** $y^2 - x^2 - y + 3x - 11 = 0$.

Find the equations of the hyperbolas satisfying the following conditions, and sketch each curve.

13. Foci at (4,0) and (10,0) and one vertex at (6,0).

14. Vertices at (3,4) and (−1,4) and eccentricity $\frac{1}{2}\sqrt{13}$.

15. Conjugate axis 12 and vertices at (−3,2) and (5,2).

16. Transverse axis 6 and foci at (2,−3) and (2,6).

17. One vertex at (3,−1), nearest focus at (5,−1) and eccentricity $\frac{3}{2}$.

18. Vertices at (2,3) and (2,8) and eccentricity 2.

19. Axes parallel to the coordinate axes and passing through the points (0,1), (0,−5), (2,−2) and (6,−2).

20. Transverse axis on the line $x - 2 = 0$, one focus at (2,−4) and having $2x - 3y - 7 = 0$ as an asymptote.

21. Vertices at (2,−1) and (2,9) and the distance from a vertex to the nearest focus equal to 6.

22. The common chord of the circles $x^2 + y^2 - 12x - 9y + 38 = 0$ and $x^2 + y^2 + 2x - 9y + 10 = 0$ as the transverse axis and eccentricity 2.

23. Center at the vertex of the parabola $x^2 + 6x - 6y + 39 = 0$, transverse axis of length 8 perpendicular to the axis of the parabola and eccentricity $\frac{5}{4}$.

24. Find the equation of the curve traced by a point which moves so that the difference of its distances from the points (2,3) and (2,−3) is always 4.

25. Find the locus of a point which moves so that its distance from the point (5,2) is always twice its distance from the line $x - 2 = 0$.

26. A point moves so that the product of its distances from the lines $x + 2y + 6 = 0$ and $x - 2y - 6 = 0$ is equal to 8. Find the equation of the locus.

27. Find the equation of the equilateral hyperbola with center at (−1,−3) and which passes through the point (5,−2).

SOME LINES ASSOCIATED WITH SECOND DEGREE CURVES

53. Introduction. In our work with curves of the second degree, we have had occasion to define certain lines such as the *radical axis*, *directrix*, *asymptote*, et cetera, which play an important part in the study of the curves. In this part of the course, we shall consider a few more lines associated with second degree curves, the most important of these being **tangents.**

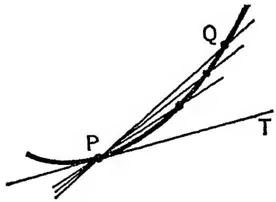


FIG. 74

54. Tangents. In general, *a tangent PT at a point P of a curve* (Fig. 74) *is defined as the limiting position of a secant PQ as Q approaches P along the curve.* This definition applies to any curve which has a tangent, and the methods of the differential calculus are used to find the slope, and hence the equation, of PT.

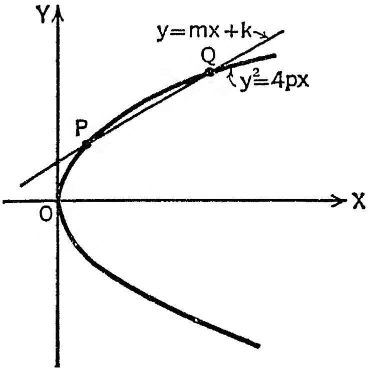


FIG. 75

As our study is limited to curves of the second degree, we are able to use a special method rather than the general calculus method mentioned above in finding equations of tangents. To make the work definite, suppose we find the equation of the line of slope m which is tangent to the parabola $y^2 = 4px$.

When a straight line $y = mx + k$ cuts the parabola y^2 in two distinct points P and Q (Fig. 75), we find the coordinates of these points by solving the equations of the line and parabola simultaneously. That is, substituting $y = mx + k$ in y^2 :

we find

$$m^2x^2 + 2mkx + k^2 = 4px,$$

or

$$m^2x^2 + 2(mk - 2p)x + k^2 = 0, \tag{38}$$

which solved for x will give the abscissas of P and Q. The ordinates may be found by substituting these values of x in $y = mx + k$.

If the two points P and Q coincide, the line is said to be tangent to the parabola. In this case, we know from our study of quadratic equations that the discriminant of equation (38) must be zero. This discriminant is $b^2 - 4ac$ for the general equation $ax^2 + bx + c = 0$ and so has the value $4(mk - 2p)^2 - 4m^2k^2$ for our equation. Solving

$$4(mk - 2p)^2 - 4m^2k^2 = 0$$

for k, we find

$$m^2k^2 - 4pmk + 4p^2 - m^2k^2 = 0, \qquad \text{or} \qquad k = \frac{p}{m}.$$

Therefore, the equation of the tangent in terms of the slope m is given by

$$y = mx + \frac{p}{m}, \tag{39}$$

a true equation for all finite values of m except $m = 0$.

In like manner, the tangents to the other second degree curves are found to be

$$y = mx \pm r\sqrt{1 + m^2}, \tag{40}$$

when the curve is the circle $x^2 + y^2 = r^2$;

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \tag{41}$$

when the curve is the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; and

$$y = mx \pm \sqrt{a^2m^2 - b^2}, \tag{42}$$

when the curve is the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

It is to be noted that these last three curves have two tangents with a given slope m, and that while a line with any finite slope may be tangent to a circle or an ellipse, this is not true for the hyperbola. In this case $a^2m^2 - b^2$ must be positive or zero in order for the tangent to be real, and thus m must have values such that $m^2 \geqq \frac{b^2}{a^2}$; that is, m cannot take values between $-\frac{b}{a}$ and $\frac{b}{a}\cdot$ When $m^2 = \frac{b^2}{a^2}$, or $m = \pm\frac{b}{a}$, the equations of the tangents are $y = \pm\frac{b}{a}x$, which we know to be the equations of the asymptotes of the hyperbola.

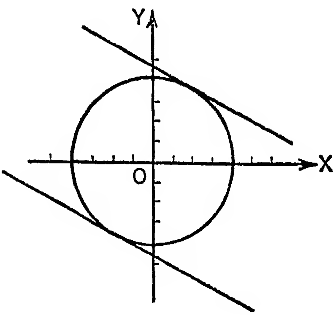


FIG. 76

EXAMPLE. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ which have the slope $-\frac{1}{2}$.

In order for the line

$$y = -\tfrac{1}{2}x + k$$

to be a tangent to the given circle (Fig. 76), the discriminant of the quadratic equation in x,

$$x^2 + (-\tfrac{1}{2}x + k)^2 = 16, \qquad \text{or} \qquad 5x^2 - 4kx + 4k^2 - 64 = 0,$$

must be zero. Therefore, $16k^2 - 20(4k^2 - 64) = 0$ and $k = \pm 2\sqrt{5}$. Hence

$$y = -\tfrac{1}{2}x \pm 2\sqrt{5}$$

are the equations of the tangents.

Equation (40) may be used as a check on the work. For $m = -\frac{1}{2}$ and $r = 4$, this equation gives $y = -\frac{1}{2}x \pm 4\sqrt{1 + \frac{1}{4}}$, or $y = -\frac{1}{2}x \pm 2\sqrt{5}$.

55. Tangent from an External Point. A method of finding the equations of tangents to a second degree curve from an external point will now be given for a special case.

Example. Find the equations of the tangents to the hyperbola

$$9x^2 - 32y^2 - 54x + 128y - 335 = 0$$

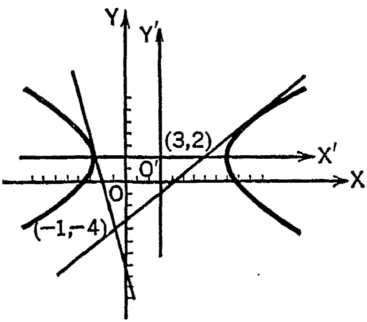


Fig. 77

from the point $(-1, -4)$.

Completing the squares, we may write the equation

$$9(x - 3)^2 - 32(y - 2)^2 = 288,$$

or, when referred to its center as origin,

$$\frac{x'^2}{32} - \frac{y'^2}{9} = 1. \qquad \text{(a)}$$

The point $(-1, -4)$ becomes $(-4, -6)$ with respect to the x' and y' axes, as $x' = x - 3$ and $y' = y - 2$. Hence our problem reduces to one of finding the equations of the tangents to (a) from the point $(-4, -6)$.

Using equation (42) with x and y replaced by x' and y', respectively, and substituting the values a^2 and b^2 from (a), we have $y' = mx' \pm \sqrt{32m^2 - 9}$ from which to obtain the tangents. Inasmuch as the lines must pass through the point $(-4, -6)$, these coordinates satisfy the equation; hence

$$-6 = -4m \pm \sqrt{32m^2 - 9}, \qquad \text{and} \qquad m = -\frac{15}{4} \text{ or } \frac{3}{4}.$$

The values of m substituted in $y' = mx' \pm \sqrt{32m^2 - 9}$ give

$$3x' - 4y' \pm 12 = 0 \qquad \text{and} \qquad 15x' + 4y' \pm 84 = 0.$$

Two of these lines, $3x' - 4y' - 12 = 0$ and $15x' + 4y' + 84 = 0$, pass through the point $(-4,-6)$ and are the required tangents referred to the x' and y' axes. To find their equations with respect to the x and y axes, we substitute $x' = x - 3$ and $y' = y - 2$, obtaining as our final results

$$3x - 4y - 13 = 0 \qquad \text{and} \qquad 15x + 4y + 31 = 0.$$

If preferred, the latter part of this example may be worked as follows. We know the slopes $-\frac{15}{4}$, $\frac{3}{4}$ and the point $(-1,-4)$, so we may use the point-slope equation of a straight line to obtain the tangents. Thus

$$y + 4 = \tfrac{3}{4}(x + 1), \qquad \text{or} \qquad 3x - 4y - 13 = 0,$$

and

$$y + 4 = -\tfrac{15}{4}(x + 1), \qquad \text{or} \qquad 15x + 4y + 31 = 0$$

are the required equations.

EXERCISES

In each of the first eleven exercises write the equation of the tangents and draw a figure.

1. To the circle $x^2 + y^2 = 9$ from the point $(2,-3)$.

2. To the hyperbola $5x^2 - 2y^2 = 18$ from the point $(1,-4)$.

3. To the ellipse $9x^2 + 16y^2 = 144$ perpendicular to the line $4x - y + 1 = 0$.

4. To the parabola $x^2 + 2x - 4y + 3 = 0$ parallel to the line $2x - 3y - 4 = 0$.

5. To the hyperbola $x^2 - 4y^2 = 16$, making equal intercepts on the coordinate axes.

6. To the circle $x^2 + y^2 - x + 2y - 29 = 0$ with a slope $m = \frac{3}{4}$.

7. To the parabola $y^2 - 4y - 9x - 5 = 0$ from the point $(-5,2)$.

8. To the circle $x^2 + y^2 + 6x - 2y - 15 = 0$ from the point $(0,6)$.

9. To the circle $x^2 + y^2 - 2x - 4y - 4 = 0$, making an angle of 135° with the x-axis.

10. To the parabola $y^2 - 4y - 8x + 2 = 0$ parallel to the line $x + 2y - 1 = 0$.

11. To the ellipse $5x^2 + 9y^2 - 36y - 9 = 0$ perpendicular to the line $3x + 4y - 13 = 0$.

12. Derive equation (40).

13. Show that the **length of a tangent** to the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

from an external point $P(x_1,y_1)$ is given by

$$t = \sqrt{x_1^2 + y_1^2 + Dx_1 + Ey_1 + F}.$$

14. Show that the *radical axis* of two circles may be defined as the locus of a point which moves in such a way that the tangents to the two circles from this point are always equal.

15. Find the slope of a line drawn from the vertex of a parabola to the point of intersection of any two tangents to the curve.

16. Derive equation (41).

17. Let any tangent to an ellipse meet the tangents at the vertices in the points A and B. Show that the product of the ordinates of A and B is equal to the square of the semi-minor axis.

18. Show that the tangents to a parabola from any point on the directrix are perpendicular.

19. A perpendicular to the axis of a parabola is erected at the focus. Show that a tangent to the curve meets this line and the directrix in points which are at the same distance from the focus.

20. If two tangents to an ellipse meet at right angles, find the locus of their point of intersection.

21. A tangent to a hyperbola is cut by the tangents at the vertices in the points M and N. Show that a circle whose center is at the midpoint of MN and which has a radius equal to one-half MN passes through the foci of the hyperbola.

22. Derive equation (42).

23. If the product of the slopes of two tangents to an ellipse is a constant, show that the locus of the point of intersection of the tangents is an ellipse or hyperbola according as the constant is positive or negative.

24. A triangle is formed by a tangent to a hyperbola and the intercepts of this line on the asymptotes. Show that the area of this triangle is a constant.

56. Tangent at a Given Point on a Second Degree Curve. To illustrate the method of finding the equation of a tangent to a curve of the second degree in terms of the coordinates of the point of contact, let us find such an equation for the circle $x^2 + y^2 = r^2$ at a point $P(x_1,y_1)$.

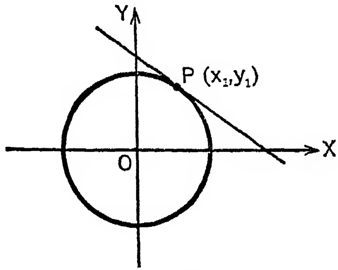


FIG. 78

In Art. 54, we found that $y = mx \pm r\sqrt{1 + m^2}$ is the equation of the tangent to the circle $x^2 + y^2 = r^2$ for all finite values of m. Now, if $P(x_1,y_1)$ is the point of tangency (Fig. 78), then (x_1,y_1) satisfies this equation of the tangent and

$$y_1 = mx_1 \pm r\sqrt{1 + m^2}.$$

Solving for m, we have

$$y_1 - mx_1 = \pm r\sqrt{1 + m^2},$$

$$y_1^2 - 2mx_1y_1 + m^2x_1^2 = r^2 + r^2m^2,$$

$$(x_1^2 - r^2)m^2 - 2x_1y_1m + y_1^2 - r^2 = 0,$$

and
$$m = \frac{2x_1y_1 \pm \sqrt{4x_1^2y_1^2 - 4(x_1^2 - r^2)(y_1^2 - r^2)}}{2(x_1^2 - r^2)}$$

$$= \frac{x_1y_1 \pm r\sqrt{x_1^2 + y_1^2 - r^2}}{x_1^2 - r^2}.$$

The point P being on the circle, its coordinates must satisfy the equation of the circle. Therefore $x_1^2 + y_1^2 = r^2$ from which we find $x_1^2 + y_1^2 - r^2 = 0$ and $x_1^2 - r^2 = -y_1^2$. On substituting these values in the expression for m, the slope of the tangent becomes

$$m = \frac{x_1y_1}{-y_1^2} = -\frac{x_1}{y_1}.$$

Knowing the slope and the point of contact, we use the point-slope equation of a straight line and find

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

which reduces to

$$y_1y - y_1^2 = -x_1x + x_1^2, \qquad \text{or} \qquad x_1x + y_1y = x_1^2 + y_1^2.$$

But $x_1^2 + y_1^2 = r^2$, and so our final equation of the tangent to the circle in terms of the coordinates of the point of contact is

$$\boldsymbol{x_1x + y_1y = r^2.} \tag{43}$$

In like manner, the equations of the tangents to the other second degree curves are found to be

$$\boldsymbol{y_1y = 2p(x + x_1),} \tag{44}$$

when the curve is the parabola $y^2 = 4px$;

$$\boldsymbol{b^2x_1x + a^2y_1y = a^2b^2,} \tag{45}$$

when the curve is the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; and

$$\boldsymbol{b^2x_1x - a^2y_1y = a^2b^2,} \tag{46}$$

when the curve is the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

Study of these equations will show that each one may be obtained from the equation of the corresponding curve by replacing x^2, y^2 and x by x_1x, y_1y and $\frac{1}{2}(x + x_1)$, respectively.

Example. Find the coordinates of a point on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, at which a tangent makes equal angles with the coordinate axes.

In Fig. 79, let $P(x_1,y_1)$ be the point whose coordinates we wish to find, and let $b^2x_1x + a^2y_1y = a^2b^2$ be the equation of the tangent at this point. By substituting $y = 0$ in this equation,

we find the x-intercept of the tangent to be $OA = \frac{a^2}{x_1}\cdot$ Similarly, by substituting $x = 0$, we have $OB = \frac{b^2}{y_1}\cdot$

By hypothesis, angle OAP is equal to angle OBP, that is,

$$\text{arc tan}\,\frac{OB}{OA} = \text{arc tan}\,\frac{OA}{OB},$$

or, replacing OA and OB by their values,

$$\text{arc tan}\,\frac{b^2}{y_1}\cdot\frac{x_1}{a^2} \qquad \text{arc tan}\,\frac{a^2}{x_1}\cdot\frac{y_1}{b^2}.$$

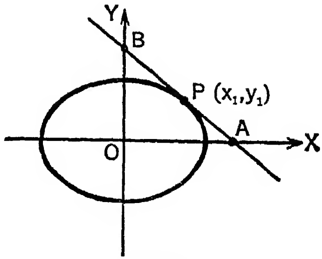


FIG. 79

Hence $\frac{b^2x_1}{a^2y_1} =$ or ${}_1^2 - a^4y_1^2 = 0$. We know also that P is on the ellipse, and, therefore, $b^2x_1^2 + a^2y_1^2 = a^2b^2$. Solving the two equations

$$b^4x_1^2 - a^4y_1^2 = 0 \qquad \text{and} \qquad b^2x_1^2 + a^2y_1^2 = a^2b^2$$

simultaneously for x_1 and y_1, we find

$$x_1 = \frac{\pm a^2}{\sqrt{a^2 + b^2}}, \qquad y_1 = \frac{\pm b^2}{\sqrt{a^2 + b^2}}$$

to be coordinates of the point P.

57. Normals. By definition *a normal to a curve at any point on the curve is a straight line which is perpendicular to the tangent at that point.* In finding the equation of the normal, it is usually best to find the equation of the tangent first; then, if m is the slope of the tangent, the equation of the normal is given by

$$y - y_1 = -\frac{1}{m}(x - x_1), \tag{47}$$

where $m \neq 0$ and (x_1,y_1) are the coordinates of the point on the curve.

Example 1. Find the equation of the normal to the ellipse $4x^2 + 9y^2 = 36$ at the point $(\sqrt{5}, \frac{4}{3})$.

By writing the equation of the tangent to the curve at any point $P(x_1,y_1)$, that is,

$$+ 9y_1y = 36,$$

and then substituting

$$x_1 = \sqrt{5},\ y_1 = \tfrac{4}{3},$$

we have

$$+ 12y = 36$$

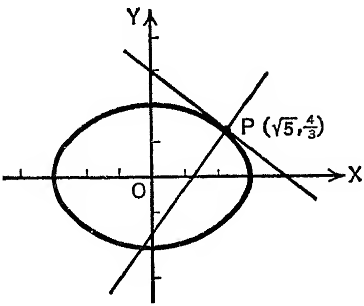


Fig. 80

as the equation of the tangent at the particular point. Since the slope of this line is $-\frac{1}{3}\sqrt{5}$, we know that the slope of the normal is $\dfrac{3}{\sqrt{5}}\cdot$ Hence the equation of the normal at the point $(\sqrt{5},\frac{4}{3})$ is

$$y - \frac{4}{3} = \frac{3}{\sqrt{5}}(x - \sqrt{5}), \qquad \text{or} \qquad 9x - 3\sqrt{5}y - 5\sqrt{5} = 0.$$

Example 2. Find the equation of the normal to the parabola $y^2 = 8x$ which is perpendicular to the line $2x - 3y + 6 = 0$.

The equation of the tangent at a point $P(x_1,y_1)$ is $y_1y = 4(x + x_1)$, or $4x - y_1y + 4x_1 = 0$. Its slope is $\dfrac{4}{y_1}$ and thus the slope of the normal is $-\frac{y_1}{4}\cdot$ But the normal is perpendicular to the line $2x - 3y + 6 = 0$. Therefore, $-\dfrac{y_1}{4} = -\dfrac{3}{2}$, or

$y_1 = 6$. Also, since P is on the curve, its coordinates satisfy the equation of the curve, that is, $y_1^2 = 8x_1$. Solving the equations $y_1 = 6$ and $y_1^2 = 8x_1$ simultaneously, we find $x_1 = \frac{9}{2}$, $y_1 = 6$ to be coordinates of the point of contact (Fig. 81). Therefore, the equation of the normal to the curve at this point is

$$y - 6 = -\tfrac{3}{2}(x - \tfrac{9}{2}),$$

or

$$6x + 4y - 51 = 0.$$

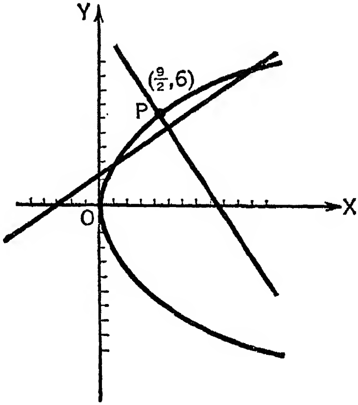


FIG. 81

EXERCISES

1. Show that $x_1x = 2p(y + y_1)$ is the equation of the tangent to the parabola $x^2 = 4py$ at the point of tangency $P(x_1,y_1)$.

2. In Fig. 82, PT is called the **length of the tangent** and PN the **length of the normal.** MT, the projection of PT on the x-axis is called the **subtangent**, and MN, the projection of PN, the **subnormal.** Find these four values for the parabola $y^2 = 4x$ when the coordinates of P are (4,4).

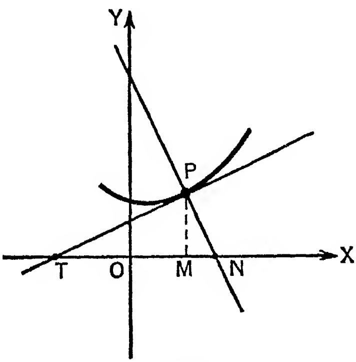


FIG. 82

3. Find the subtangent and subnormal for the hyperbola $5x^2 - 12y^2 = 60$ if the point of tangency is $(-6\sqrt{2},5)$.

4. Prove that the subnormal of a parabola is equal to one-half the latus rectum.

5. Prove that the vertex of a parabola bisects the subtangent.

6. Given that the slope of $Ax^2 + By^2 + Dx + Ey + F = 0$ is $m = -\dfrac{2Ax_1 + D}{2By_1 + E}$ at the point

(x_1,y_1), show that

$$Ax_1x + By_1y + \tfrac{1}{2}D(x + x_1) + \tfrac{1}{2}E(y + y_1) + F = 0$$

is the equation of its tangent. It should be observed that such a proof enables us to find the equation of a tangent to a second degree curve of this type by substituting x_1x, y_1y, $\frac{1}{2}(x + x_1)$ and $\frac{1}{2}(y + y_1)$ for x^2, y^2, x and y, respectively, in the equation of the curve.

7. Find the equations of the tangent and normal at the point (1,1) on the ellipse $x^2 + 4y^2 + 4x - 24y + 15 = 0$.

8. Find the equations of the tangent and normal at the point (6,7) on the hyperbola $x^2 - y^2 - 6x + 4y + 21 = 0$.

9. Find the lengths of the subtangent and subnormal for the ellipse $9x^2 + y^2 - 54x - 10y + 81 = 0$ when the point of tangency is (2,1).

10. Find the equations of the normals to the hyperbola

$$16x^2 - 25y^2 + 64x + 50y - 105 = 0$$

which are parallel to the line $x - y - 5 = 0$.

11. Find the equations of the tangents to the parabola $y^2 = -8x$ at the ends of the latus rectum. Show that these tangents are perpendicular to each other and that they intersect on the directrix.

12. Show that the tangent at any point P of a parabola makes equal angles with the line joining P to the focus and a line through P parallel to the axis of the parabola.

13. Prove that the tangent to a parabola at one end of the latus rectum is parallel to the normal at the other end.

14. The **slope of a curve** at a point on the curve is the slope of the tangent at that point. Two curves are said to be **orthogonal** at a point of intersection if their tangents meet at right angles at that point. Show that the circles $x^2 + y^2 - 2x + y + 1 = 0$ and $x^2 + y^2 - 4x + 3 = 0$ are orthogonal.

15. Show that $D_1D_2 + E_1E_2 = 2F_1 + 2F_2$ is the condition that the two circles $x^2 + y^2 + D_1x + E_1y + F_1 = 0$ and

$$x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

be orthogonal. (Hint: Draw the right triangle formed by joining the centers of the circles and the point of contact and use the Pythagorean theorem.)

16. Find the equation of the circle orthogonal to the three circles:

$$x^2 + y^2 - 6 = 0, \qquad x^2 + y^2 + 2x - y - 4 = 0$$

and

$$x^2 + y^2 + 4x - 8y + 1 = 0.$$

(Hint: Let $x^2 + y^2 + Dx + Ey + F = 0$ be the equation of the required circle and find D, E, F by using the condition derived in Exercise 15.)

17. Show that the ellipse $7x^2 + 16y^2 = 112$ and the hyperbola $5x^2 - 4y^2 = 20$ are orthogonal and confocal. (**Confocal** means having foci in common.)

18. Show that the normal to an ellipse at a given point P bisects the angle formed by the lines joining this point to the foci.

19. Show that in general an ellipse and a hyperbola which are confocal are also orthogonal.

20. If the ordinates of three points on a parabola are in geometrical progression, show that the tangents at the extreme points will meet on the ordinate of the middle point.

21. A tangent drawn to a hyperbola at a point P on the curve intersects the conjugate hyperbola in the points M and N. Show that P is the mid-point of MN.

THE CONICS

58. Historical Background. The curves which we have been studying as locus problems, that is, the circle, the parabola, the ellipse and the hyperbola, are all members of a class of curves called **conics, or conic sections,** and may be obtained by passing a plane through a right circular cone. Special cases arise when the cutting plane passes through the vertex of the cone; these, however, will not be considered in our discussion. In Fig. 83, if the cutting plane is perpendicular to the axis of the cone, or parallel to the base, we have a **circle,** such as C. If the plane is parallel to an element of the cone, say BV, we have a **parabola,** such as P. If the plane meets both elements AV and BV, we have an **ellipse,** such

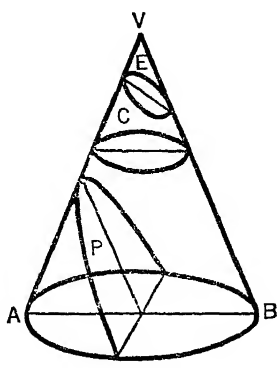


FIG. 83

as E. When the cutting plane meets an element BV produced, or cuts both nappes of the cone, as in Fig. 84, we have a **hyperbola.**

It is to be observed that when the plane of the ellipse is parallel to the base of the cone, the curve is a circle. Hence the circle is a special case of the ellipse, and the definition of a conic to be given in the next article includes it only as such.

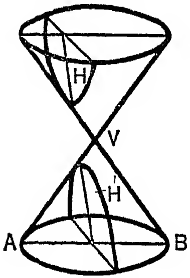


FIG. 84

We have mentioned (Art. 1) that as early as the third century before Christ, Apollonius of Perga obtained the three sections, *parabola, ellipse* and *hyperbola,* by passing a plane through a cone, but we have not yet considered the work that had been done in this field prior to his time. The discovery of the conics was to some extent a consequence of attempts to find the solution of one of "the three famous problems of antiquity." These problems were: (a) *squaring a circle,* that is, to construct a square such that the area will be equal to that of a given circle; (b) *trisecting an angle,* that is, to divide an angle into three equal parts; (c) *duplicating a cube,* that is, to construct a cube such that the volume will be double that of a given cube. The tools used in solving the problems were restricted to the straightedge and compasses, which, we know, dooms one to failure from the beginning. The ancients did solve the problems by other means, however, and their investigations opened up new fields of study.

In the fifth century before Christ, Hippocrates of Chios reduced the problem of duplicating the cube to one of finding two mean proportionals in continued proportion. That is, if the means of two lines of lengths a and $2a$ are x and y, we may write in our present-day notation $a/x = x/y = y/2a$. The first proportion gives $x^2 = ay$ and the second, $y^2 = 2ax$. By squaring the first of these equations and substituting the value of y^2 in the second, we obtain $x^3 = 2a^3$. Therefore, the mean propor-

tional x is the edge of a cube whose volume is twice that of a given cube of edge a.

We know that the two equations, $x^2 = ay$ and $y^2 = 2ax$, represent parabolas and that it is only necessary to find their point of intersection (Fig. 85) to obtain the desired mean proportionals, x and y. Thus, while the problem is an easy one for us, we must remember that the parabola had not been discovered at the time of Hippocrates and that his instruments, ruler and compasses, made the task of constructing the mean proportionals an impossible one. Though he failed to find a geometric construction for the means, his work was very important in that it turned the attention of later mathematicians in this direction, thus contributing to the discovery of the conics.

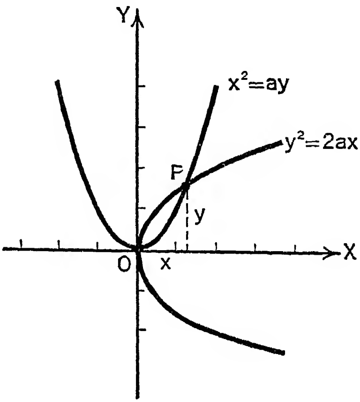


FIG. 85

Credit for the discovery of the conic sections is given to Menæchmus, who was born about 375 B.C. and was a member of the school founded by Plato at Athens. He used three cones: one with an *acute angle* at the vertex, another with a *right angle* at the vertex, and the third with an *obtuse angle* at the vertex. By intersecting each cone in turn by a plane perpendicular to an element, he obtained the three sections which we now call *ellipse, parabola,* and *hyperbola.* These names were not used, however, until the time of Apollonius about a century later; in the meantime they were called sections of the *acute-angled, right-angled* and *obtuse-angled* cone, or *Menæchmian triads.* Knowing the conics, Menæchmus was able to construct the mean proportionals mentioned above and thus, by disregarding the original restrictions, solve the problem of duplicating the cube.

Great strides were made in the study and development of the conics during the next century, and at the end of that period Apollonius crowned these efforts with his great work, *Conic Sections*. Apollonius generalized the work of his predecessors by using a scalene cone, and, by varying the angle of inclination of the cutting plane, obtained the three sections from a single cone. His work contains a discussion of many of the properties of conics and, as we know, it held full sway in this field until the sixteenth century when the analytical methods we use today were first employed.

59. Definition and Equation of a Conic. Instead of defining a conic as the section of a cone and discussing its properties as the Greeks did, we define such a curve as the path traced by a moving point and use algebraic methods in our discussion. Thus, *a conic is the locus of a point which moves in such a way that its distance from a fixed point is in constant ratio to its distance from a fixed straight line.* The fixed point is called the **focus**, the fixed line the **directrix** and the constant ratio the **eccentricity** of the conic.

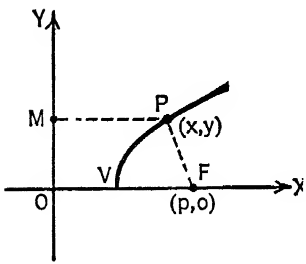


FIG. 86

Taking the y-axis as the fixed line (Fig. 86), a point $F(p,0)$ on the x-axis as the fixed point, and designating the constant ratio by e, our equation may be written

$$FP = eMP,$$

where P is any point on the curve and $e > 0$. In terms of coordinates, this equation becomes

$$\sqrt{(x - p)^2 + y^2} = ex,$$

which may be reduced to the form

$$(1 - e^2)x^2 + y^2 - 2px + p^2 = 0. \tag{48}$$

Examining the equation in the light of our knowledge of second degree curves, we see that the type of conic it represents depends upon the value of e. If $e = 1$, the coefficient of x^2 is zero and the equation represents a *parabola*. When $e < 1$, the coefficients of the squared terms are both positive but of different magnitudes and we have an *ellipse*. When $e > 1$, $1 - e^2$ is negative and our curve is a *hyperbola*. To simplify the discussion of the above equation, let us assume that p is positive throughout and treat each of the three cases separately.

CASE I. When $e = 1$, the equation reduces to $y^2 = 2p\left(x - \frac{p}{2}\right)$, and we see that the x-axis is the axis of the parabola and that the vertex is at the point $\left(\frac{p}{2}, 0\right)$, half-way between the focus and directrix. We note also that the condition $FP = eMP$ reduces to $FP = MP$, the definition of a parabola given in Art. 38.

CASE II. $e < 1$. Equation (48) may be written

$$x^2 - \frac{2p}{1 - e^2}x + \frac{y^2}{1 - e^2} = -\frac{p^2}{1 - e^2},$$

or, by completing the square,

$$\left(x - \frac{p}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{p^2e^2}{(1 - e^2)^2}.$$

Translating the origin to the point $\left(\frac{p}{1 - e^2}, 0\right)$, which is the center of the conic, our equation becomes

$$x'^2 + \frac{y'^2}{1 - e^2} = \frac{p^2e^2}{(1 - e^2)^2}. \qquad (49)$$

If $y' = 0$, we find $x' = \pm\frac{pe}{1 - e^2}$ and hence the coordinates of

the vertices are $\left(\frac{\pm pe}{1 - e^2},0\right)$ with respect to O' as origin (Fig. 87). Likewise, if $x' = 0$, we have $y' = \frac{\pm pe}{\sqrt{1 - e^2}}\cdot$ Denoting the lengths of the major and minor axes by $2a$ and $2b$, respectively, we may write

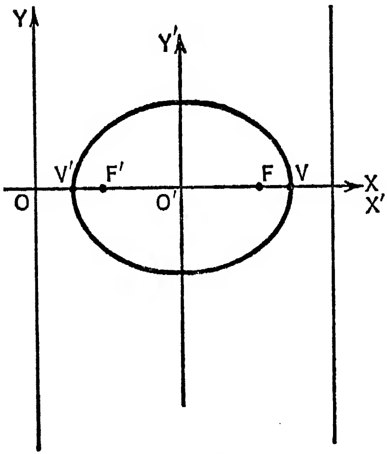


FIG. 87

$$a = \frac{pe}{1 - e^2}, \quad b = \frac{pe}{\sqrt{1 - e^2}}$$

and $$a^2 = \frac{p^2e^2}{(1 - e^2)^2},$$

$$b^2 = \frac{p^2e^2}{1 - e^2} = a^2(1 - e^2).$$

Making these substitutions, equation (49) becomes

$$x'^2 + \frac{a^2}{b^2}y'^2 = a^2,$$

or

$$b^2x'^2 + a^2y'^2 = a^2b^2.$$

If we let $c^2 = a^2 - b^2$, as in our former work with the ellipse, we have $c^2 = a^2 - a^2(1 - e^2) = a^2e^2$ and, therefore, $c = ae$, or $e = \frac{c}{a}\cdot$

Since the curve is symmetric about the center, there are two directrices, each at a distance $1 - \quad = \frac{a}{e}$ from O'. Their equations are $x' = \pm\frac{a}{e}\cdot$ The foci are at a distance $c = ae$ from the center and hence their coordinates are $(\pm ae,0)$. Substituting $x' = ae$ in $b^2x'^2 + a^2y'^2 = a^2b^2$, we obtain $y'^2 = b^2(1 - e^2) = \frac{b^4}{a^2}$

since $1 - e^2 = \frac{b^2}{a^2}\cdot$ Therefore $y' = \pm\frac{b^2}{a}$ and the length of the latus rectum is $\frac{2b^2}{a}\cdot$

CASE III. $e > 1$. In this case also the equation of the conic,

$$(1 - e^2)x^2 + y^2 - 2px + p^2 = 0,$$

reduces to the form

$$x'^2 + \frac{y'^2}{1 - e^2} = \frac{p^2e^2}{(1 - e^2)^2},$$

or

$$x'^2 - \frac{y'^2}{e^2 - 1} = \frac{p^2e^2}{(e^2 - 1)^2},$$

the center being at the point

$$\left(\frac{p}{1 - e^2},0\right).$$

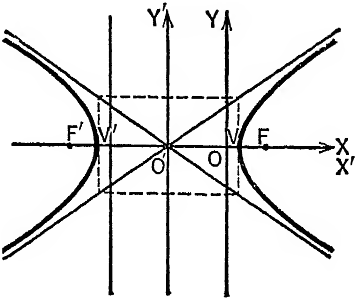


FIG. 88

Since $1 - e^2$ is negative for the hyperbola and p was assumed to be positive, the center of the curve falls to the left of the y-axis (Fig. 88). For $a = \frac{pe}{e^2 - 1}$ and $b = \frac{pe}{\sqrt{e^2 - 1}}$, the last of the above equations reduces to the form

$$x'^2 - \frac{a^2}{b^2}y'^2 = a^2, \qquad \text{or} \qquad b^2x'^2 - a^2y'^2 = a^2b^2.$$

Letting $c^2 = a^2 + b^2$, we again find that $c = ae$, and hence the coordinates of the foci are $(\pm ae,0)$. Likewise, the coordinates of the vertices are $\left(\frac{\pm pe}{e^2 - 1},0\right)$, the equations of the directrices are $x' = \pm\frac{a}{e}$ and the length of the latus rectum is $\frac{2b^2}{a}\cdot$ The equations of the asymptotes are found by factoring $b^2x'^2 - a^2y'^2 = 0$.

The ellipse and hyperbola are called **central conics** since the curves are symmetric about a finite point called the center. This is not true of the parabola.

60. Applications. Since a detailed discussion of many of the scientific applications of the conics requires a knowledge of the calculus, we shall content ourselves with a list of some of the uses of these curves, indicating the basic theorems in a few cases.

The cable of a suspension bridge uniformly loaded along the horizontal hangs in a parabolic arch. The path of a projectile fired at an angle with the horizontal is a parabola, if air resistance is neglected. Arches in buildings and bridges are often parabolic in shape. Parabolic reflectors and reflecting telescopes make use of parabolic mirrors; such a mirror being formed by revolving a parabola about its axis.

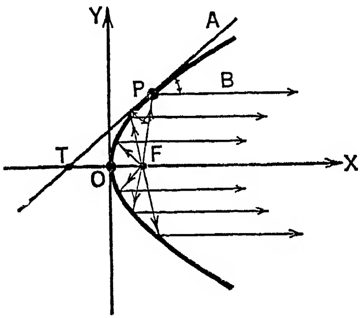


Fig. 89

The basic theorem of this last application is given in Exercise 12, page 120. There, we proved that angle TPF in Fig. 89 is equal to angle APB. Hence light emanating from a source placed at F will strike the parabolic surface and be reflected in parallel rays, giving a beam of light which may be controlled by turning the mechanism. This is the principle used in the design of headlights, searchlights, etc.

The same type of mirror is used in reflecting telescopes, where now the rays coming from a distant source strike the mirror in parallel lines and are collected at the focus. The design of a burning-glass is based on this theorem also. In this case, the rays from the sun strike the convex surface of the glass and, after passing through it, are collected at the focus on the other side. In fact, it is thought that the word *focus* was coined from this

use of the parabola since the Latin meaning of the word is *hearth* or *fireplace*.

The "Ptolemaic Theory" of the universe, advanced by Claudius Ptolemy between 125 A.D. and 151 A.D., placed the *earth* at the center with the sun and the planets revolving around this body. This theory was held until the sixteenth century when the studies of Copernicus, Kepler and others were made. In 1543, Copernicus completed his work on celestial orbits in which he claimed that the planets revolve about the *sun* as a center. He retained the postulate of uniform circular motion, however, and this caused his results to be at variance with some of the known facts about planetary motion. Kepler (1571–1630) held to this same postulate for a period, but after much computation, he concluded that the planets move in *ellipses* with the sun at one focus. This was confirmed by later astronomers and mathematicians, including Newton, who showed that the law of gravitation conforms to such a theory. It is now known that many comets, as well as the planets, have elliptical orbits.

Ellipses are also used in architecture and bridge design. The Colosseum at Rome is in the shape of an ellipse, and many beautiful stone and concrete bridges have elliptical arches. The design of whispering galleries is based on Exercise 18, page 121, where a sound from one focus may be heard at the other, but is inaudible between these two points. Elliptical gears are used in machines such as power punches, planers, etc., where a slow but powerful stroke is required.

A hyperbola referred to its asymptotes as axes may be used to express Boyle's law of a perfect gas. This equation is also used in the study of economics. Locating a source of sound (range finding) is accomplished by means of hyperbolas, as illustrated below.

Example. Sound recording instruments are placed at two points A and B which are 300 feet apart, Fig. 90. A sound coming from P is heard 0.20 of a second earlier at B than at A. Locate the position of P.

Take the axes as in the figure, where the origin is the mid-point of AB. The coordinates of A and B are, therefore, $(-150,0)$ and $(150,0)$, respectively. Since sound travels approximately 1100 feet per second, P is $0.2(1100) = 220$ feet nearer B than A. Hence $AP - BP = 220$, and if the coordinates of P are (x,y), we have

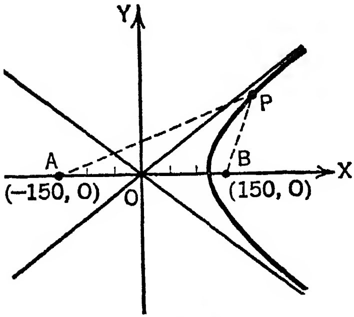


FIG. 90

$$\sqrt{(x+150)^2+y^2} - \sqrt{(x-150)^2+y^2} = 220,$$

or

$$104x^2 - 121y^2 = 1{,}258{,}400.$$

Therefore P is on a branch of a hyperbola represented by this equation. By recording the sound at two other points, a second hyperbola could be drawn through P, thus locating the point at the intersection of the two curves.

EXERCISES

1. The distance between the towers of a suspension bridge is 1500 feet and the vertex of the parabolic arc of the cable is 130 feet below the points of support. Find the equation of the curve which the cable makes when the origin is at the vertex.

2. The major axis of the earth's orbit is approximately 185.8 million miles and the eccentricity is about $\frac{1}{60}$. Find the equation of the earth's path and its greatest and least distance from the sun.

3. The Colosseum at Rome is approximately 615 feet long and 510 feet wide. Find the eccentricity and the equation of the ellipse.

4. Four sound recording instruments, A, B, C, D, are placed on two mutually perpendicular lines, A and B on one line and C and D on the other. The distance between A and B is one mile while that between C and D is 3000 yards. The point of intersection O of the lines is at the mid-point of AB and of CD. A sound is recorded 2 seconds earlier at B than at A, and 3 seconds earlier at C than at D. Set up the equations for locating the source of the sound.

61. Theorems and Locus Problems. In proving general theorems and solving locus problems, the axes should be so chosen that the work involved is reduced to a minimum.

EXAMPLE. The vertices V', V of an ellipse are joined to a point P on the perimeter by straight lines. Two other lines are drawn through P perpendicular to $V'P$ and VP, respectively. Prove that the segment intercepted by this second pair of lines on the axis of the ellipse is equal to the latus rectum.

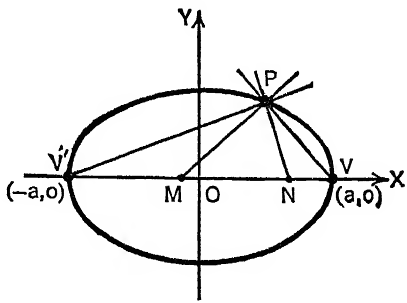


FIG. 91

Choose the origin at the center of the ellipse and let the major axis, of length $2a$, coincide with the x-axis. If the minor axis is $2b$, the equation of the ellipse is $b^2x^2 + a^2y^2 = a^2b^2$ and the coordinates of V' and V are $(-a,0)$ and $(a,0)$, respectively (Fig. 91).

If the coordinates of P are (x_1,y_1), the slope of $V'P$ is $\frac{y_1}{x_1 + a}$, and the slope of VP is $\frac{y_1}{x_1 - a}$. Hence NP, drawn perpendicular to $V'P$, has a slope $-\frac{x_1 + a}{y_1}$ and its equation becomes, by means of the point-slope form,

$$y - y_1 = -\frac{x_1 + a}{y_1}(x - x_1),$$

or $$(x_1 + a)x + y_1y - ax_1 - x_1^2 - y_1^2 = 0.$$

In like manner, the equation of MP is

$$y - y_1 = -\frac{x_1 - a}{y_1}(x - x_1),$$

or $$(x_1 - a)x + y_1y + ax_1 - x_1^2 - y_1^2 = 0.$$

To find the x-intercepts N and M, substitute $y = 0$ in each of the above equations and solve for x. Thus

$(x_1 + a)x - ax_1 - x_1^2 - y_1^2 = 0$ and $x = \dfrac{x_1^2 + y_1^2 + ax_1}{x_1 + a}$

is the distance ON; $(x_1 - a)x + ax_1 - x_1^2 - y_1^2 = 0$ and $x = \dfrac{x_1^2 + y_1^2 - ax_1}{x_1 - a}$ is the distance OM. Since $MN = MO + ON = ON - OM$, we have

$$MN = \frac{x_1^2 + y_1^2 + ax_1}{x_1 + a} - \frac{x_1^2 + y_1^2 - ax_1}{x_1 - a} = -\frac{2ay_1^2}{x_1^2 - a^2}.$$

The point $P(x_1,y_1)$ is on the ellipse; therefore $b^2x_1^2 + a^2y_1^2 = a^2b^2$ and $y_1^2 = \dfrac{b^2}{a^2}(a^2 - x_1^2)$. Substituting this value of y_1^2, we obtain

$$MN = -\frac{2a}{x_1^2 - a^2} \cdot \frac{b^2}{a^2}(a^2 - x_1^2) = \frac{2b^2}{a},$$

the length of the latus rectum.

EXERCISES

1. Show that the minor axis of an ellipse is a mean proportional between the major axis and the latus rectum.

2. Perpendiculars to the x-axis are drawn from points on the circumference of the circle $x^2 + y^2 = r^2$. Find the locus of the mid-points of the perpendiculars.

3. Show that the abscissa of a point on the parabola $x^2 = 4py$ is a mean proportional between the ordinate of that point and the latus rectum.

4. A perpendicular is dropped from a focus of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ to an asymptote. Find the distance from the foot of this perpendicular to the center of the curve.

5. Show that the product of the distances from any point on a hyperbola to its asymptotes is a constant.

6. Find the equation of the hyperbola of eccentricity $\frac{3}{2}$, whose center is at the vertex of the parabola $y^2 - 6y - 3x + 18 = 0$, and one of whose directrices coincides with the directrix of the parabola.

7. The ends of the base of a triangle are at $(\pm a,0)$. Find the locus of the third vertex when the median from this vertex to the base of the triangle is a mean proportional between the other two sides of the triangle.

8. If the product of the slopes of the variable sides of the triangle in Exercise 7 is $\frac{b^2}{a^2}$, find the locus of the third vertex.

9. For any hyperbola, show that an asymptote, a line through a focus perpendicular to the asymptote, and the directrix nearest this focus meet in a point.

10. Find the perpendicular distance from a focus of the hyperbola $b^2y^2 - a^2x^2 = a^2b^2$ to one of its asymptotes.

11. Show that the product of the distances between directrices and between the latera recta of an ellipse is equal to the square of the major axis.

12. Find the equation of the parabola with vertex at the origin which passes through the extremities of the right-hand latus rectum of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

13. If the volume of a given cube is 8 cubic yards, construct the length of the edge of a cube of twice this volume.

14. Prove that the lines drawn from the intersection of the axis and directrix of a parabola to the ends of the latus rectum are perpendicular.

15. Find the equation of the ellipse of eccentricity $\frac{1}{3}$, whose center is at the intersection of the lines $x + y = 3$ and $2x - y = 2$, and which has the line $x + 5 = 0$ as a directrix.

16. Show that the distance from the center of a rectangular hyperbola to any point P on the curve is a mean proportional between the distances $F'P$ and FP, where F' and F are the foci of the hyperbola.

17. If two conjugate hyperbolas have eccentricities e_1 and e_2, respectively, show that $e_1^2 + e_2^2 = e_1^2e_2^2$.

18. Show that the segment of an asymptote which is bounded by the directrices of a hyperbola is equal to the transverse axis.

19. A line drawn through a focus F of an equilateral hyperbola parallel to an asymptote meets the curve at a point P. Find the length of FP.

20. An ellipse and a hyperbola have their foci and latera recta in common. Find the relationship between their eccentricities.

21. A perpendicular from a point Q on the transverse axis of a hyperbola meets the curve at a point P and an asymptote at R. Show that $(RQ)^2 - (PQ)^2$ is always constant.

22. Call the vertex of a parabola V and let P and Q be two points on the curve. If the lines PV and QV intersect at right angles, show that PQ cuts the axis of the parabola in a fixed point.

23. A line joins the vertex V of a parabola to a point P on the curve. A second line PQ intersects the first one at right angles. If Q is on the axis of the parabola, show that the projection of PQ on this axis is of the same length as the latus rectum.

24. A tangent is drawn to an ellipse at a point P on the curve. A line joining P and the center of the ellipse intersects a directrix at Q. Show that a line passing through the corresponding focus and perpendicular to the tangent will pass through Q also.

25. A straight line cuts a hyperbola in the points P and Q, and its asymptotes in the points M and N. Show that the mid-points of PQ and MN coincide.

26. The points P and Q of an ellipse whose vertices are V' and V are so situated that the chord PQ is perpendicular to the major axis. Find the locus of the point of intersection of PV' and QV.

27. Lines drawn from the vertices V' and V of an equilateral hyperbola intersect at a point P on the curve. Show that the bisector of the angle $V'PV$ is parallel to an asymptote.

28. Through each of two points P and Q of a hyperbola, lines are drawn parallel to both asymptotes. Show that the center of the curve lies on one of the diagonals of the parallelogram thus formed.

CHAPTER V

THE GENERAL EQUATION OF THE SECOND DEGREE

62. Rotation of Axes. In Chapter II we have considered the transformation of coordinates known as **translation,** in which the coordinate axes are moved into a new position parallel to the original position. We shall now study another transformation of the coordinates known as **rotation,** in which the origin remains fixed and the axes are turned through some fixed angle about this point.

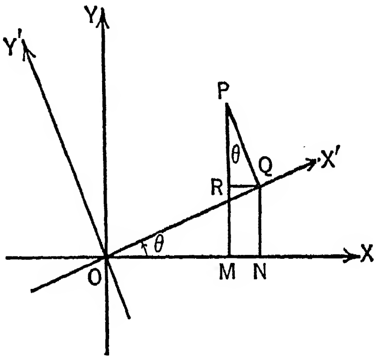


FIG. 92

Let P be any point whose coordinates referred to the original axes OX and OY are (x,y) and referred to the new axes OX' and OY' are

Then in Fig. 92 $OM = x$, $MP = y$, $OQ = x'$, and $QP = y'$. Let the angle of rotation XOX' be represented by θ. Draw RQ perpendicular to PM, and QN perpendicular to OX. Then angle $RPQ = \theta$. From the figure, we have

$$x = OM = ON - MN = ON - RQ,$$

and
$$y = MP = MR + RP = NQ + RP.$$

But $ON = OQ \cos\theta$, $RQ = PQ \sin\theta$, $NQ = OQ \sin\theta$, and $RP = PQ\cos\theta$. By substitution, we get

$$\mathbf{x = x'\cos\theta - y'\sin\theta,}$$

and
$$\mathbf{y = x'\sin\theta + y'\cos\theta.}$$

By the use of these **formulas of rotation,** the equation of any locus referred to one pair of rectangular axes may be transformed into the equation of the same locus referred to a second pair of axes which pass through the same origin and make an angle θ with the original axes.

EXERCISES

Transform each of the following equations by means of a rotation of axes through the indicated acute angle θ.

1. $x^2 - y^2 = 25$, $\theta = 45°$. **2.** $x^2 + y^2 = 25$, $\theta = 30°$.

3. $x + y - 5 = 0$, $\theta = 45°$.

4. $2x - 3y + 12 = 0$, $\theta = \tan^{-1} \frac{2}{3}$.

5. $x^2 + 4xy + y^2 = 4$, $\theta = 45°$. **6.** $xy = 18$, $\theta = 45°$.

7. $x^2 - 2xy + 2y^2 = 1$, $\theta = \tan^{-1} 2$.

8. $3x^2 + 3xy - y^2 = 1$, $\theta = \tan^{-1} \frac{1}{3}$.

9. $2x^2 + 24xy - 5y^2 = 8$, $\theta = \cos^{-1} \frac{4}{5}$.

10. $41x^2 - 84xy + 104y^2 = 500$, $\theta = \tan^{-1} \frac{1}{2}$.

11. Find the coordinates of the point (1,4) after the axes have been rotated through an angle of 45°.

12. Remove the y-term from $x^2 + y^2 - 3x - 3y = 0$ by rotating the axes.

13. Transform the equation $x^2 - 2xy + y^2 - 8\sqrt{2}x - 8\sqrt{2}y = 0$ by means of a rotation of axes through an angle of 45°.

14. Transform the equation $x^2 - y^2 = 9$ by a rotation to a new set of axes which bisect the angles between the original axes.

15. Transform the equation $x^2 - 3y^2 + 1 = 0$ by means of a rotation of axes through an angle of 30°.

16. Transform the equation $x^2 + xy + y^2 - x - y + 3 = 0$ by means of a rotation of axes through an angle of 45°.

63. Transformation of the General Equation of the Second Degree by Rotation. The general equation of the second degree in two variables may be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad \text{(a)}$$

We shall now show that this equation may be transformed into an equation without an xy-term by a proper rotation of the axes to which the locus of the equation is referred. Let us substitute for x and y the values given by the rotation formulas,

$$x = x' \cos \theta - y' \sin \theta,$$

$$y = x' \sin \theta + y' \cos \theta.$$

By this substitution equation (a) becomes

$$\begin{aligned} A(x' \cos \theta - y' \sin \theta)^2 &+ B(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) \\ &+ C(x' \sin \theta + y' \cos \theta)^2 + D(x' \cos \theta - y' \sin \theta) \\ &+ E(x' \sin \theta + y' \cos \theta) + F = 0. \end{aligned} \tag{b}$$

In this new equation, after expanding and collecting all $x'y'$-terms, we have as the coefficient of $x'y'$

$$B' = -2A \sin \theta \cos \theta + B \cos^2 \theta - B \sin^2 \theta + 2C \sin \theta \cos \theta.$$

From trigonometry,

$$2 \sin \theta \cos \theta = \sin 2\theta, \qquad \text{and} \qquad \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

Using these values, we may write the coefficient of $x'y'$ as

$$B' = -(A - C) \sin 2\theta + B \cos 2\theta.$$

Now to remove the $x'y'$-term it is necessary that a value of θ be so determined that this coefficient shall become zero. This condition gives $B' = 0$ when

$$-(A - C) \sin 2\theta + B \cos 2\theta = 0;$$

that is,

$$\tan 2\theta = \frac{B}{A - C}, \tag{51}$$

where $A - C \neq 0$. In case $A - C = 0$, $\cos 2\theta = 0$ and hence $\theta = 45°$. Since there is always a value of 2θ between 0° and 180° for which equation (51) is true, there is always a positive

acute angle θ through which the axes may be rotated so as to remove the $x'y'$-term. If the axes are rotated through this particular angle, the general equation (a) above will be transformed into

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F = 0, \tag{c}$$

where the new coefficients, collected from the expanded form of (b), are as follows:

$$\begin{aligned} A' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta, \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta, \\ D' &= D\cos\theta + E\sin\theta, \\ E' &= E\cos\theta - D\sin\theta. \end{aligned}$$

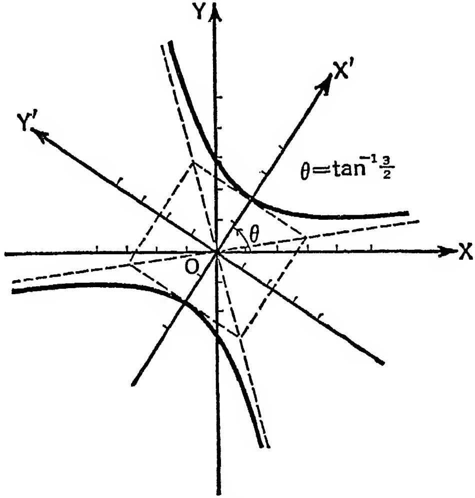


FIG. 93

EXAMPLE. Remove the xy-term from the equation

$$16x^2 - 108xy - 29y^2 + 260 = 0.$$

Using (51), we have

$$\tan 2\theta \quad \frac{-108}{16+29} \quad -12$$

To find the functions of θ, we may either construct a figure for 2θ or use the formula

$$\cos 2\theta = \frac{1}{\sec 2\theta} = -\frac{1}{\sqrt{1+\tan^2 2\theta}} = -\frac{1}{\sqrt{1+\frac{144}{25}}} = -\frac{5}{13}.$$

By selecting the sign of $\cos 2\theta$ to be the same as that of $\tan 2\theta$, the angle 2θ will be in the second quadrant, and therefore θ will be acute. Also,

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}, \qquad \text{and} \qquad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}.$$

Substituting the value of $\cos 2\theta$ in these expressions, we get

$$\sin\theta = \frac{3}{\sqrt{13}}, \qquad \text{and} \qquad \cos\theta = \frac{2}{\sqrt{13}}.$$

The formulas for rotating the axes are, therefore,

$$x = \frac{2x' - 3y'}{\sqrt{13}}, \qquad \text{and} \qquad y = \frac{3x' + 2y'}{\sqrt{13}}.$$

Substituting these values in the equation of the curve, we obtain

$$16\left(\frac{2x'-3y'}{\sqrt{13}}\right)^2 - 108\left(\frac{2x'-3y'}{\sqrt{13}}\right)\left(\frac{3x'+2y'}{\sqrt{13}}\right) - 29\left(\frac{3x'+2y'}{\sqrt{13}}\right)^2 + 260 = 0.$$

By expanding and collecting like terms, we get

$$845x'^2 - 676y'^2 = 3380, \qquad \text{or} \qquad 5x'^2 - 4y'^2 = 20.$$

Fig. 93 is drawn by first constructing the new axes making an angle θ with the original axes, where $\theta = \tan^{-1}\frac{3}{2}$, and then drawing the hyperbola on the OX' and OY' reference frame.

64. The Characteristic of the General Second Degree Equation. The expression $\Delta = B^2 - 4AC$ is called the **characteristic**

of the general equation of the second degree and we shall now prove that it remains unchanged in value by rotation. An expression, such as this, that remains unchanged in value after a transformation is said to be **invariant** under that transformation. Let us take the values of A', B', and C' as given in Art. 63 and substitute them in the expression $B'^2 - 4A'C'$. After expanding and collecting like terms, we obtain

$$\begin{aligned} &B^2 \cos^4\theta + B^2 \sin^4\theta - 8AC \sin^2\theta\cos^2\theta - 4AC\sin^4\theta \\ &\qquad\qquad\qquad\qquad -4AC\cos^4\theta + 2B^2\sin^2\theta\cos^2\theta \\ &\quad = (B^2 - 4AC)\cos^4\theta + (B^2 - 4AC)\sin^4\theta \\ &\qquad\qquad\qquad\qquad + 2(B^2 - 4AC)\sin^2\theta\cos^2\theta \\ &\quad = (B^2 - 4AC)(\sin^2\theta + \cos^2\theta)^2 = B^2 - 4AC. \end{aligned}$$

Therefore $B'^2 - 4A'C' = B^2 - 4AC$, or *the characteristic is invariant under the transformation of rotation.*

It may also be shown that the coefficients of the second degree terms are invariant under translation, that is, when we make the substitutions $x = x' + h$ and $y = y' + k$. We may state, therefore, that *the characteristic is invariant under the transformations of translation and rotation.*

EXERCISE. Prove that the characteristic is invariant under translation. Prove, also, that $A + C$ is invariant under both translation and rotation.

65. The Locus of the General Second Degree Equation. We have considered the transformation of rotation and have shown that the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{(a)}$$

may be transformed into

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0 \qquad \text{(b)}$$

by a general rotation. Also, if we choose a value of θ such that

$$\tan 2\theta = \frac{B}{A - C}, \qquad (A \neq C)$$

the $x'y'$-term will be removed and the new equation will become

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F = 0. \tag{c}$$

Furthermore, we have shown that in this case

$$\Delta = B^2 - 4AC = -4A'C', \qquad \text{since} \qquad B' = 0.$$

In our study we have developed the following facts in regard to the final form of the general second degree equation shown by (c):

1. If $\Delta < 0$, then A' and C' have the same sign, and the locus of the equation is an *ellipse* when $A' \neq C'$.
2. If $\Delta > 0$, then A' and C' have different signs, and the locus is a *hyperbola*.
3. If $\Delta = 0$, then either $A' = 0$ or $C' = 0$, and the locus is a *parabola*.

Therefore, we may say that the general second degree equation will represent a parabola, an ellipse or a hyperbola depending upon the value of the characteristic. Exceptional cases may occur in which the second degree equation can be factored into two first degree factors and hence represent straight lines.

EXAMPLE. Identify the conic $16x^2 - 24xy + 9y^2 + 20x - 140y - 300 = 0$, simplify the equation and draw the curve, showing each set of axes.

We find that $\Delta = B^2 - 4AC = (-24)^2 - 4(16)(9) = 0$, and therefore the conic is a parabola. To remove the xy-term, we rotate the axes through the angle θ determined by

$$\tan 2\theta = \frac{B}{A - C} = \frac{-24}{16 - 9} = -\frac{24}{7}.$$

Hence, $\cos 2\theta = -\frac{7}{25}$, and we find that

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5},$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}.$$

The rotation formulas become

$$x = \frac{3x' - 4y'}{5}, \qquad y = \frac{4x' + 3y'}{5}.$$

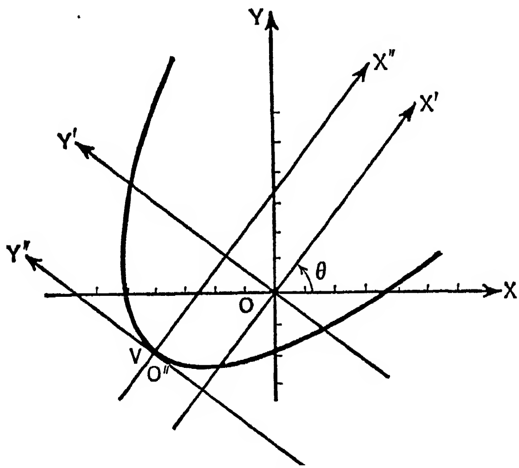


FIG. 94

Substituting these values in the equation, we have

$$16\left(\frac{3x'-4y'}{5}\right)^2 - 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right) + 9\left(\frac{4x'+3y'}{5}\right)^2$$

$$+ 20\left(\frac{3x'-4y'}{5}\right) - 140\left(\frac{4x'+3y'}{5}\right) - 300 = 0.$$

After expanding, collecting like terms and reducing, we obtain

$$y'^2 - 4x' - 4y' - 12 = 0.$$

This represents a parabola with axis parallel to the OX'-axis (Fig. 94). We find the vertex of the parabola by completing the square of the y'-terms, getting

$$(y' - 2)^2 = 4(x' + 4).$$

The coordinates of the vertex are $(-4,2)$ with reference to the OX' and OY' axes, and hence we may write the equation with reference to axes drawn through this point parallel to the OX' and OY' axes as

$$y''^2 = 4x''.$$

The figure is drawn by first constructing the OX' and OY' axes at an angle θ with the original axes, where $\theta = \tan^{-1}\frac{4}{3}$, and then drawing $O''X''$ and $O''Y''$ parallel to OX' and OY' through the point $x' = -4$, $y' = 2$. The parabola is then drawn on the last set of axes.

EXERCISES

Identify the following conics, simplify the equations[1] and draw the curve, showing each set of axes. Check the figure by finding the intercepts on the original axes.

1. $5x^2 + 2xy + 5y^2 - 12x - 12y = 0.$
2. $x^2 + 4xy + y^2 - x - y + 4 = 0.$
3. $x^2 - 2xy + y^2 - 8x + 16 = 0.$
4. $4x^2 + 4xy + y^2 + 8x - 16y = 0.$
5. $12xy - 5y^2 + 48y - 36 = 0.$

[1] When $\triangle = B^2 - 4AC = 0$, that is, when the conic is a *parabola*, the axes should be rotated first, because it is impossible to remove both linear terms by a translation. When $\triangle \neq 0$, however, translation may be followed by rotation. The student should satisfy himself of the truth of this statement by translating the axes of the general equation of the second degree and examining the values of h and k.

6. $2x^2 + 2xy + 2y^2 - 6y - 9 = 0.$

7. $x^2 + 3xy + y^2 + x - y - 1 = 0.$

8. $2xy + 4x - 6y + 1 = 0.$

9. $16x^2 - 24xy + 9y^2 - 60x - 80y + 400 = 0.$

10. $25x^2 - 14xy + 25y^2 + 142x - 178y + 121 = 0.$

11. $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0.$

12. $x^2 -$ [illegible] $+$ [illegible] $- 6x + 2y = 0.$

13. $x^2 +$ [illegible] $^2 + 4x + 48y + 34 = 0.$

14. $x^2 - 4xy -$ [illegible] $+ 10x + 4y = 0.$

15. $3x^2 + 2xy$ [illegible] $^2 - 8x - 8y = 0.$

16. $2x^2 - 8xy - 4y^2 - 4y + 1 = 0.$

17. $3x^2 + 12xy + 8y^2 - 5 = 0.$

18. $36x^2 + 24xy + 29y^2 - 72x + 126y + 81$ [illegible] $0.$

CHAPTER VI

EQUATIONS IN OTHER FORMS

66. Introduction. In the previous chapters we have discussed equations of the first, second, and higher degrees and have used rectangular coordinates to show their corresponding graphs. In this chapter we shall consider other equations which lend themselves to this method of plotting. In addition, a new system of coordinates, called **polar,** will be defined, and it will be seen that many equations assume a simpler form when expressed in terms of such coordinates.

67. Parametric Equations. When each of the rectangular coordinates x and y is expressed in terms of a third variable, the two resulting equations taken together constitute the **parametric equations** of a curve. The third variable is called the **parameter.** To plot a curve represented by a set of parametric equations, a table is computed by assigning values to the parameter and computing corresponding values of x and y. The coordinates (x,y) are then plotted and the curve drawn in the usual manner.

Example 1. Plot the curve represented by the parametric equations

$$x = 2t^2, \qquad y = \tfrac{1}{2}t^3.$$

A table of values is given and the curve is plotted in Fig. 95. To illustrate the method used in computing the coordinates (x,y) of any point let us assign a value to the parameter t, say $t = 2$. This gives $x = 2(2)^2 = 8$ and $y = \frac{1}{2}(2)^3 = 4$. The other coordinates are computed in a similar manner.

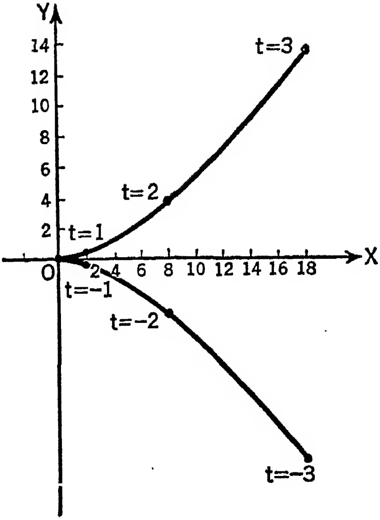


FIG. 95

t	x	y
0	0	0
1	2	0.5
2	8	4.0
3	18	13.5
−1	2	− 0.5
−2	8	− 4.0
−3	18	−13.5

NOTE: The parameter does not appear on the graph, but the plotted points may be marked by the corresponding values

The equation of a curve may be changed from the parametric form to the rectangular form by eliminating the parameter. The method used depends upon the example. For the illustration given above, we may solve the first equation for t, getting $t = \left(\frac{x}{2}\right)^{\frac{1}{2}}$, and this value substituted for t in the second equation will give

$$y = \tfrac{1}{2}\left(\frac{x}{2}\right)^{\frac{3}{2}}.$$

This may be simplified and expressed in the form

$$32y^2 = x^3.$$

EXAMPLE 2. Find the rectangular equation of the curve

$$x = a \cos \varphi, \qquad y = b \sin \varphi.$$

Here the parameter is the variable angle φ. To eliminate the parameter in this case, we use the trigonometric identity

$\sin^2 \varphi + \cos^2 \varphi = 1$. Let us divide the first equation by a, the second by b, square both equations and add. This will give

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the rectangular equation of an ellipse.

The parametric equations of an ellipse $x = a \cos \varphi$, $y = b \sin \varphi$ are associated with another method of constructing such a curve. This method makes use of *auxiliary circles* and is given below. Construct two concentric circles with centers at the origin and with radii a and b, $(a > b)$ as in Fig. 96. Through O draw any line making an angle φ with the x-axis and meeting the circles in points R and Q. Through R draw NR parallel to OX, and through Q draw QM parallel to OY. Let these lines intersect in the point $P(x,y)$. Draw RS perpendicular to OX. Then from the figure, we have

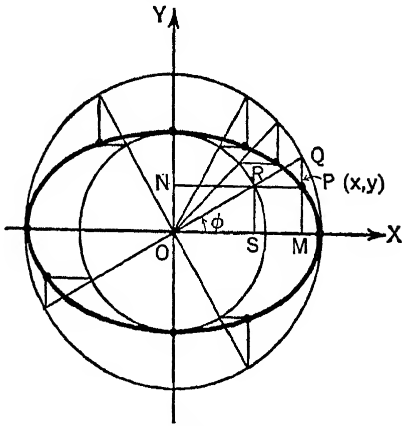


FIG. 96

$$\frac{OM}{OQ} = \cos \varphi \qquad \text{and} \qquad \frac{RS}{OR} = \sin \varphi.$$

But $OM = x$, $OQ = a$, $RS = y$, and $OR = b$. Therefore $\frac{x}{a} = \cos \varphi$ and $\frac{y}{b} = \sin \varphi$. These are the parametric equations of the ellipse, and hence the point P lies on this curve. The two circles have as their diameters the major and minor axes of the ellipse and are called the *auxiliary circles* of the ellipse. By

giving different values to φ, that is by drawing OQ at different angles with the x-axis, we may construct as many points as necessary to draw the ellipse. The angle φ is called the *eccentric angle* of the point P.

EXERCISES

Draw the following curves using the parametric equations. Also find the rectangular equation for each curve.

1. $x = t,\ y = t + 2.$

2. $x = \frac{1}{2}t,\ y = t^2.$

3. $x = 2t,\ y = \dfrac{3}{t}$

4. $x = 2\cos\theta,\ y = 2\sin\theta.$

5. $x = 3\cos\theta,\ y = 4\sin\theta.$

6. $x = t + 5,\ y = t^2 - 2.$

7. $x = t^3,\ y = t^2.$

8. $x = t,\ y = t^3.$

9. $x = \frac{1}{2}t - 4,\ y = t^2.$

10. $x = 3t,\ y = 2\sqrt{1 - t^2}.$

11. $x = 2\sec\theta,\ y = 2\tan\theta.$

12. $x = \cos t,\ y = \sin t\cos t.$

13. $x = t^2 - 1,\ y = t^2.$

14. $x = 2\sin t,\ y = 2(1 - \cos t).$

15. $x = \frac{1}{2}t^2,\ y = \frac{1}{3}t^3.$

16. $x = 4\sin\theta + \cos\theta,\ y = \cos\theta.$

17. $x = t - t^2,\ y = t + t^2.$

18. $x = t^2 - 4t,\ y = 1 - t^2.$

19. $x = a\cos^3\theta,\ y = a\sin^3\theta.$

20. $x = a\cos^4\theta,\ y = a\sin^4\theta.$

Trace the following curves, without finding their rectangular equations.

21. $x = \frac{1}{3}\sin\theta,\ y = \sin 2\theta.$

22. $x = 4\cos 2\theta,\ y = 2\sin^2\theta.$

23. $x(1 + t^2) = 2t^2 - 2,\ y(1 + t^2) = 2t^3 - 2t.$

24. $x(1 + t^3) = 3at,\ y(1 + t^3) = 3at^2.$

25. $x = 5(\theta - \sin\theta),\ y = 5(1 - \cos\theta).$

26. $x = 8\cos\theta + 4\cos 2\theta,\ y = 8\sin\theta - 4\sin 2\theta.$

27. In mechanics it is shown that the path of a projectile fired with an initial velocity v_0 at an angle of elevation θ is given by the parametric equations

$$x = v_0 \cdot \cos\theta \cdot t, \qquad y = v_0 \cdot \sin\theta \cdot t - 16t^2.$$

a. Identify the curve by finding its rectangular equation.
b. By finding the x-intercept, show that the horizontal range is given by $\dfrac{v_0^2 \sin 2\theta}{32}$.
c. Show that the maximum ordinate is $\dfrac{v_0^2 \sin^2\theta}{64}$.
d. What angle of elevation will give the maximum range?

28. Using the equation given in Exercise 27, find the equation of the path of a projectile, and the range, under the following conditions:
a. Initial velocity 500 feet per second, $\theta = 45°$.
b. Initial velocity 800 feet per second, $\theta = 30°$.
c. Initial velocity 1700 feet per second, $\theta = 20°$.

68. Exponential and Logarithmic Curves. An **exponential** curve has an equation of the form $y = a^x$, where x and y are variables and a is a constant such that $a > 0$. Since, by the definition of a logarithm, this equation may be written $x = \log_a y$, the locus may also be called a **logarithmic** curve.

x	y
0.1	-1.000
0.5	-0.301
1	0.000
2	0.301
3	0.477
4	0.602
5	0.699
6	0.778
7	0.845
8	0.903
9	0.954
10	1.000

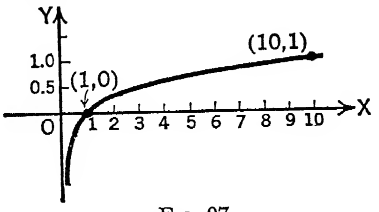


FIG. 97

EXAMPLE 1. Plot the curve $y = \log_{10} x$; also the curve $y = 10^x$.

To draw the curve $y = \log_{10} x$, the values shown above may be taken from any table of common logarithms and plotted as in Fig. 97.

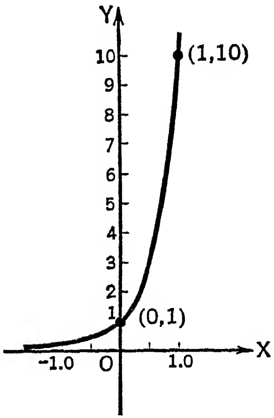


FIG. 98

Now $y = \log_{10} x$ may be written $10^y = x$. If we interchange the variables, we have

$$y = 10^x,$$

which represents the same curve as the one drawn above but with the axes as indicated in Fig. 98.

EXAMPLE 2. Plot the curve $y = \log_3 x$.

We may write the equation in the form $3^y = x$, and then, by taking the common logarithms of both sides, obtain

$$\log_{10} 3^y \qquad \log_{10} x, \qquad \text{or} \qquad y \log_{10} 3 = \log_{10} x.$$

Therefore, $$y = \frac{1}{\log_{10} 3} \cdot \log_{10} x = \frac{1}{\ } \log_{10} x = 2.09 \log_{10} x.$$

But $y = \log_3 x$. Hence $\log_3 x = 2.09 \log_{10} x$, and we may use a table of common logarithms to find the values of $\log_3 x$ for any value of x.

EXERCISES

Plot the following curves by computing a table of values.

1. $y = \log_2 x$. (Use $\log_2 x = \frac{1}{\log_{10} 2} \cdot \log_{10} x = 3.32 \log_{10} x$.)

2. $y = 2^x$.

3. $y = \log_e x$, where $e = 2.71828 \ldots$. (Use natural logarithm tables if available; if not use

$$\log_e x = \frac{1}{\log_{10} e} \cdot \log_{10} x = 2.302 \log_{10} x.)$$

4. $y = (1 + x)^{\frac{1}{x}}$. (Use logarithms to compute values.)

5. $y = 4^{x+2}$.

6. $y = e^x$.

7. $y = 5 \cdot 2^{-x}$.

8. $y = 2(e^x + e^{-x})$.

69. Trigonometric Curves. Corresponding to every trigonometric function there is a trigonometric curve. The method of drawing these curves is essentially the same as that used in plotting the graphs of algebraic functions. As an illustration, let us consider the **sine curve** $y = \sin x$. Before drawing the graph by means of plotted points, a discussion of the equation will show some important characteristics of the curve. The y-intercept is $y = 0$ and the x-intercepts are $x = n\pi$, where n is any positive or negative integer, or zero. The curve is symmetrical about the origin since $-y = \sin(-x) = -\sin x$. Since, by definition, y cannot exceed 1 in numerical value, the curve must lie entirely between the lines $y = 1$ and $y = -1$. The maximum value of y is called the **amplitude** of the curve. The curve is **periodic** since the sine function repeats its values when the angle is increased by 360° or 2π radians. The period is 2π since $\sin(x + 2\pi) = \sin x$, and, therefore, it is only necessary to plot values of x such that $0 \leqq x \leqq 2\pi$.

The curve is drawn in Fig. 99 by plotting the points computed in the table, and using the information given in the discussion above.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	0.5	0.87	1.0	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

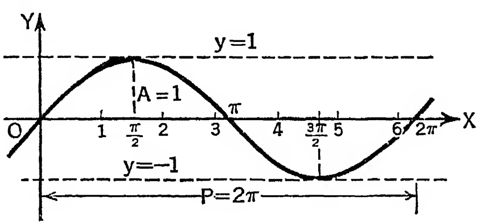


FIG. 99

Although sine curves may have different periods and amplitudes, they all have the same general appearance. It is im-

portant to note that the maximum numerical value of y occurs at the first and third quarter periods, and that the curve may be repeated by sliding it along the x-axis one period.

EXAMPLE 1. Find the amplitude and period of $y = 1.5 \sin 2x$, and sketch the curve.

The maximum value of y occurs when $\sin 2x = 1$, or the amplitude is 1.5. To find the period, we observe that $\sin(2x + 2\pi) = \sin 2(x + \pi) = \sin 2x$, or the period is π

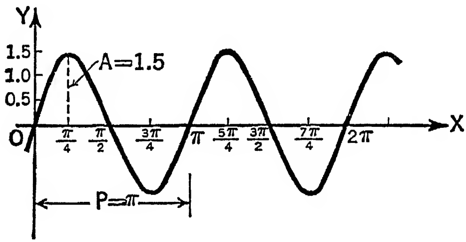


FIG. 100

radians. To sketch the curve, we first lay off the total period of π radians on the x-axis and divide it into quarter periods as indicated in Fig. 100. Then the amplitude is measured at the odd quarter periods. Additional points may be plotted if a more accurate figure is desired.

The period and amplitude of the *cosine curve* and the sine curve are the same, and the curves have the same appearance. The other trigonometric curves are plotted in a similar manner as the sine curve. It is sometimes desirable to use degrees as units on the horizontal axis if only the general shape of the curve is desired, but radian measure must be used in any case where the equation of a curve contains both algebraic and trigonometric terms.

70. Graphing by Composition of Ordinates. When the equation of a curve consists of several terms, it is sometimes convenient to consider each term as representing a curve and obtain

the complete curve by adding the ordinates of the simpler curves. The curves must be plotted on the same axes and to the same scale and then the original curve is drawn as in the following illustration.

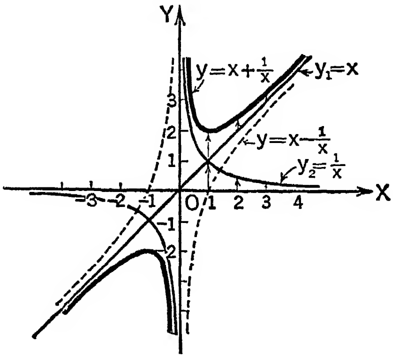


FIG. 101

EXAMPLE 1. Plot the graph of the equation

$$y = x + \frac{1}{x}.$$

Consider the graphs of the two equations

$$y_1 = x \qquad \text{and} \qquad y_2 = \frac{1}{x}.$$

Evidently an ordinate of the given curve corresponding to any given abscissa is the sum of the ordinates y_1 and y_2, since

$y = y_1 + y_2 = x + \frac{1}{x}$. To obtain the points of the given curve, we select any value of x and add the corresponding ordinates. For example, at $x = 2$, $y_1 = 2$ and $y_2 = \frac{1}{2}$, and, therefore, $y = 2 + \frac{1}{2}$, or $\frac{1}{2}$ unit is added graphically to the ordinate of the line $y_1 = x$, as indicated on the figure.

The dotted curve represents the graph of $y = x - \dfrac{1}{x}$ and is obtained by subtracting the ordinates of the curve $y_2 = \dfrac{1}{x}$ from those of the line $y_1 = x$.

EXAMPLE 2. Plot the graph of the equation $y = \dfrac{x}{2} + \dfrac{3}{2}\sin\dfrac{\pi x}{5}$.

On the same axes and to the same scale, we first plot the graphs of $y_1 = \dfrac{x}{2}$ and $y_2 = \dfrac{3}{2}\sin\dfrac{\pi x}{5}$. The ordinates of the two curves are added as indicated in Fig. 102 to give the graph of the original equation. In plotting the graph of the second curve, it

should be noted that it is a sine curve with amplitude $\frac{3}{2}$ and period $(2\pi) \div \frac{\pi}{5} = 10$ units. Particular attention must be given to the algebraic sign of the ordinates in adding.

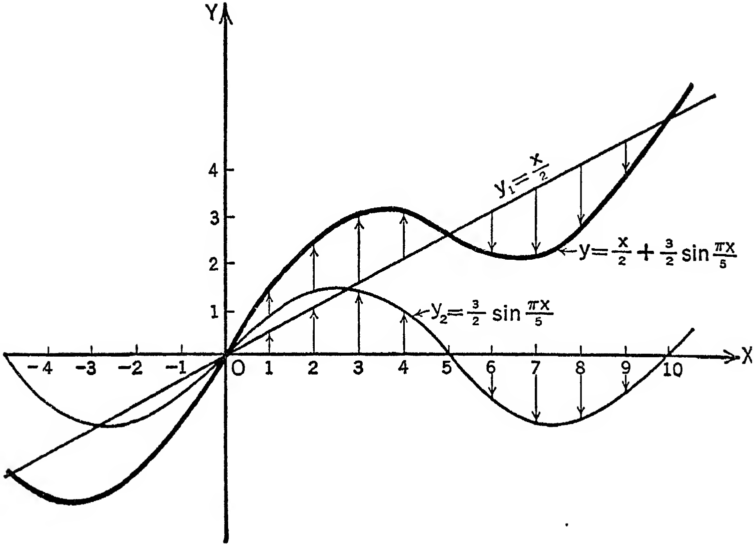


FIG. 102

EXERCISES

Plot the following curves by computing a table of values.

1. $y = 2 \sin x.$

2. $y = 1.5 \sin \dfrac{\pi x}{3}.$

3. $y = \frac{1}{2} \cos 2x.$

4. $y = \sin(x + 30°).$

5. $y = 2 \cos \left(x + \dfrac{\pi}{4}\right).$

6. $y = \tan x$ and $y = \cot x$ on the same axes.

7. $y = \sec x$ and $y = \csc x$ on the same axes.

Sketch the following periodic curves after finding the period and amplitude.

8. $y = \sin 2x$.

9. $y = 2.5 \cos x$.

10. $y = 4 \sin 3x$.

11. $y = \dfrac{1}{2} \sin \dfrac{\pi x}{2}$.

12. $y = \dfrac{3}{2} \sin \dfrac{\pi x}{4}$.

13. $y = 5 \cos \dfrac{x}{2}$.

Draw the following curves by the addition of ordinates.

14. $y = \dfrac{x}{3} + \dfrac{1}{3} \cos x$.

15. $y = \sin x + \cos x$.

16. $y = e^x - \sin 2x$.

17. $y = \frac{1}{2}(e^x + e^{-x})$.

18. $y = \sin x + \sin 2x$.

19. $y = \frac{1}{2} \sin x + 2 \cos x$.

71. Polar Coordinates. *In polar coordinates the position of a point is determined by a direction and a distance* rather than by two distances, as in rectangular coordinates, and *the frame of reference consists of a point and a directed line.* Thus, in Fig. 103, let O be a fixed point, called the *origin*, or **pole**, and OX be a fixed directed line, called the *initial line*, or **polar axis.** The position of any point P is determined by two numbers, the angle $XOP = \theta$ and the distance $OP = r$. The coordinate r is called the **radius vector** and θ the **vectorial angle.** The usual convention of signs used in trigonometry applies to the vectorial angle, in which a positive angle is generated by a counter-clockwise rotation and a negative angle by a clockwise rotation of the initial side. The radius vector r is positive when it is measured from the pole along the terminal side of the angle, and negative when measured in the opposite direction. *To plot a point, the angle is first drawn in the proper direction, thus locating the terminal side, and then the distance r is measured either along the terminal side, if positive, or along the terminal side produced through the pole, if negative.*

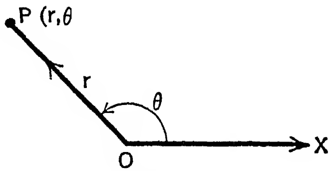


FIG. 103

It should be noticed that *one* pair of polar coordinates will determine one, and only one, point of the plane, but that any given point may have an unlimited number of polar coordinates. If the angle is restricted to values between 0° and 360°,

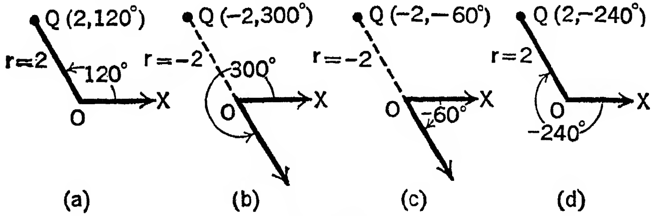


FIG. 104

any given point may be designated by *four* different pairs of polar coordinates as illustrated in Fig. 104, where the points (2,120°), (−2,300°), (−2,−60°), and (2,−240°) all locate the same point Q.

72. Plotting in Polar Coordinates. The equation of a curve in polar coordinates is a relation between the vectorial angle θ and the radius vector r. To plot a curve, the equation is solved for r in terms of θ and a table of values constructed by assigning values to θ and computing the corresponding values of r. Usually intervals of 10° to 20° will be sufficiently close to draw the curve, but the table of values should include values of θ from 0° to 360° unless the curve possesses symmetry, in which case an extensive table is unnecessary. As an illustration, a table has been computed for $r = 2(1 - \cos\theta)$ and the graph

θ	$\cos\theta$	$1 - \cos\theta$	r
0°	1.000	0.000	0.00
15°	0.966	0.034	0.07
30°	0.866	0.134	0.27
45°	0.707	0.293	0.59
60°	0.500	0.500	1.00
75°	0.259	0.741	1.48
90°	0.000	1.000	2.00
105°	−0.259	1.259	2.51
120°	−0.500	1.500	3.00
135°	−0.707	1.707	3.41
150°	−0.866	1.866	3.73
165°	−0.966	1.966	3.93
180°	−1.000	2.000	4.00

drawn in Fig. 105. The student should check the values in the table and locate each point on the curve.

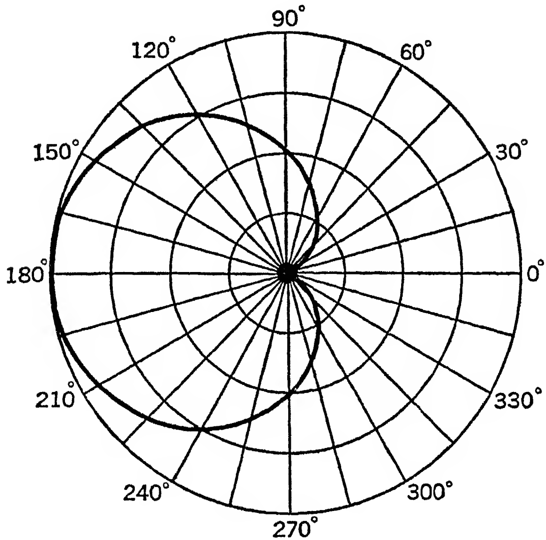


FIG. 105

Because of the symmetry of the cosine function it is unnecessary to compute values of θ from 180° to 360°.

EXERCISES

1. Plot the following points, using the same pole and polar axis: (5,30°), (−3,15°), (−5,−20°), (8,−60°), (2,90°), (−8,270°), (−8,−40°), (10,0°), (−8,90°), (−3,−180°).

2. Write three other pairs of polar coordinates for each of the points: (2,30°), (3,150°), (−4,300°), (−5,−240°).

3. One vertex of an equilateral triangle is at (4,0°). If the center of the triangle is at the pole, find a set of polar coordinates for each of the remaining vertices.

4. Where is a point whose radius vector is 5? whose vectorial angle is 45°? Express each of these conditions in the form of an equation.

Construct a table of values and plot the following curves.

5. $r \cos \theta = 5.$
6. $r \csc \theta = 2.$
7. $r = 2 \cos \theta.$
8. $r = -4 \sin \theta.$
9. $r = 2 \cos \theta + 4 \sin \theta.$
10. $r(1 - \cos \theta) = 6.$
11. $r(1 + 2 \sin \theta) = 5.$
12. $r(2 - \cos \theta) = 4.$
13. $4r = \sec \theta \tan \theta.$
14. $r = \tan \theta.$
15. $r = \cot \theta.$
16. $r = 1 + 2 \sin \theta.$ (Limaçon.)
17. $r = 2(1 - \sin \theta).$ (Cardioid.)
18. $r = 3 + 3 \cos \theta.$ (Cardioid.)
19. $r^2 = 4 \cos 2\theta.$ (Lemniscate.)
20. $r = 2\theta.$ (Spiral of Archimedes.)
21. $(r - 2)^2 = 16\theta.$ (Parabolic Spiral.)
22. $r = 8 \sin 2\theta.$ (Four-leaved rose.)
23. $r = 10 \sin 3\theta.$ (Three-leaved rose.)
24. $r = 6 \sin 4\theta.$ (Eight-leaved rose.)
25. $r = 5 \tan^2 \theta \sec \theta.$ (Semi-cubical parabola.)

73. Relations between Polar and Rectangular Coordinates. If the pole coincides with the origin of rectangular coordinates and the polar axis OX is taken as the positive x-axis as shown in Fig. 106, then any point P may be considered as having rectangular coordinates (x,y) and polar coordinates (r,θ). The relations between the two systems may be taken directly from the figure. Thus

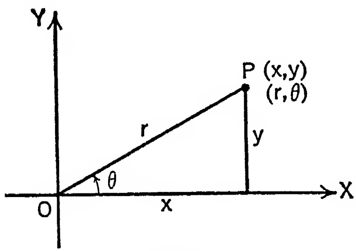


FIG. 106

$$x = r \cos \theta, \qquad y = r \sin \theta;$$
$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1} \frac{y}{x}. \tag{52}$$

By means of these relations, we are able to transform an equation in polar coordinates to one in rectangular coordinates, and vice versa. Such transformations will be illustrated in the following examples.

EXAMPLE 1. Transform the equation $r = 5 \cos \theta$ into rectangular coordinates.

Substituting $r = \sqrt{x^2 + y^2}$ and $\cos \theta = \frac{x}{r}$ from the formulas above, we get $\sqrt{x^2 + y^2} = \frac{5x}{\sqrt{x^2 + y^2}}$, or $x^2 + y^2 - 5x = 0$.

EXAMPLE 2. Transform $(3 - 2 \cos \theta)r = 2$ into rectangular coordinates.

The equation may be written $3r - 2r \cos \theta = 2$, and then, by substituting $r = \sqrt{x^2 + y^2}$ and $r \cos \theta = x$, we have

$$3\sqrt{x^2 + y^2} - 2x = 2.$$

By transposing $-2x$, squaring both sides, and combining terms, we have, finally,

$$5x^2 + 9y^2 - 8x - 4 = 0.$$

EXAMPLE 3. Transform $r = 4 \sin 2\theta$ into rectangular coordinates.

By using $\sin 2\theta = 2 \sin \theta \cos \theta$, we may write the equation as $r = 8 \sin \theta \cos \theta$. Then by substituting values of $\sin \theta$ and $\cos \theta$, we have $r = 8\frac{y}{r} \cdot \frac{x}{r}$, or $r^3 = 8xy$. But $r = \sqrt{x^2 + y^2}$, and hence $(x^2 + y^2)^{\frac{3}{2}} = 8xy$. Finally, we obtain

$$(x^2 + y^2)^3 = 64x^2y^2.$$

EXAMPLE 4. Transform $x^2 + 2y^2 = 8$ into polar coordinates.

Substituting $x = r \cos \theta$ and $y = r \sin \theta$ we obtain

$$r^2 \cos^2 \theta + 2r^2 \sin^2 \theta = 8,$$

which may be reduced to $r^2(1 + \sin^2 \theta) = 8$.

74. Polar Equations of the Straight Line. To derive the equation of a line in polar coordinates, consider Fig. 107 in which L is any line in the plane which does not pass through the pole.

Let $ON = p$ be the perpendicular distance from O to N. Then the polar coordinates of N are (p,ω). Let P be any point on the line L, having polar coordinates (r,θ). Then angle $NOP = \theta - \omega$ and, from right triangle ONP, we have

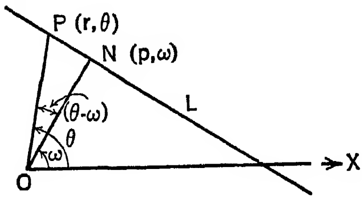


FIG. 107

$$OP \cos(\theta - \omega) = ON,$$

or

$$r \cos(\theta - \omega) = p. \qquad (53)$$

Since this formula is true for any point P on the line, it is the equation of the line.

The following special cases should be considered:

(a) If the line L is *perpendicular to the polar axis*, either $\omega = 0°$ or $\omega = 180°$ and equation (53) may be written

$$r \cos\theta = \pm p. \qquad (54)$$

(b) If the line L is *parallel to the polar axis*, either $\omega = 90°$ or $\omega = 270°$, and the equation becomes

$$r \sin\theta = \pm p. \qquad (55)$$

(c) If the line L passes through the pole, the vectorial angle is the same for every point on the line, and the equation may be written $\theta = \alpha$, where α is a constant angle.

75. Polar Equations of Circles. In Fig. 108 let (r_1,θ_1) be the coordinates of the center of a circle of radius k and let a point on the circumference be $P(r,\theta)$. Then, by the Cosine Law, we have

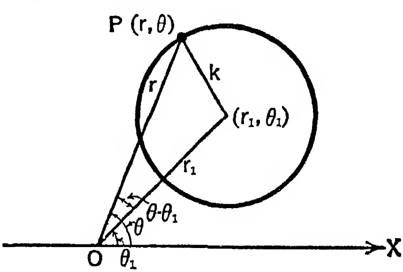


FIG. 108

$$k^2 = r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1). \qquad (56)$$

If (r_1,θ_1) is at the pole, we have $r_1 = 0$, and the above equation becomes

$$r = \pm k, \tag{57}$$

which is the equation of a circle of radius k with center at the pole.

If $\theta_1 = 0°$ and $k = \pm r_1$, equation (56) reduces to

$$r = 2k \cos \theta \qquad \text{or} \qquad r = -2k \cos \theta \tag{58}$$

which are equations of circles passing through the pole and having centers on the polar axis and this axis produced through the pole, respectively.

If $\theta_1 = 90°$ and $k = \pm r_1$, equation (56) reduces to

$$r = 2k \sin \theta \qquad \text{or} \qquad r = -2k \sin \theta, \tag{59}$$

which are equations of circles passing through the pole and having centers on the $90°$ axis.

If $r_1 = k$, equation (56) reduces to

$$r = a \cos \theta + b \sin \theta, \tag{60}$$

the equation of a circle passing through the pole and having polar and $90°$ intercepts of a and b, respectively.

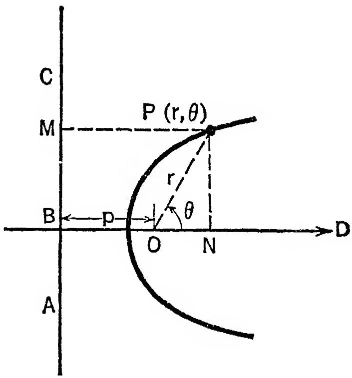


FIG. 109

76. Polar Equations of the Conics. Using the definition of a conic given in Chapter IV, namely, *a conic is the locus of a point which moves in such a way that its distance from a fixed point is proportional to its distance from a fixed line*, we may derive its equation in polar coordinates as follows. In Fig. 109, let ABC be the fixed line

and O the fixed point. Then for $P(r,\theta)$, the variable point, we have

$$\frac{OP}{MP} = e, \qquad \text{or} \qquad OP = e \cdot MP,$$

where e, the eccentricity, is the proportionality constant. To express this equation in terms of polar coordinates r and θ, let O be the pole and OD the polar axis. Then $OP = r$ and

$$MP = BO + ON = p + r \cos \theta.$$

Hence we have $$r = e(p + r \cos \theta),$$

or, solving for r, $$r = \frac{ep}{1 - e \cos \theta}. \tag{61}$$

The value of e will determine which of the three conics (parabola, ellipse or hyperbola) we may have in any special case. In order to show this, let us transform equation (61) into rectangular coordinates. We get

$$r = \frac{ep}{1 - e \cdot \frac{x}{r}} = \frac{epr}{r - ex},$$

which reduces to $$r = e(p + x).$$

Since $r = \sqrt{x^2 + y^2}$, we have $\sqrt{x^2 + y^2} = e(p + x)$,

which becomes $x^2 + y^2 = e^2p^2 + 2e^2px + e^2x^2$,

and finally $$(1 - e^2)x^2 + y^2 = e^2p(p + 2x). \tag{62}$$

If $e = 1$, equation (62) becomes $y^2 = p(p + 2x)$, which is the equation of a *parabola*.

If $e < 1$, the coefficient of x^2 is positive and the curve is an *ellipse*.

If $e > 1$, the coefficient of x^2 is negative and the curve is a *hyperbola*.

In the derivation of equation (61) the polar axis was taken as the principal axis of the conic. If the 90° axis is the principal axis, $\cos\theta$ is replaced by $\sin\theta$ in (61), and we get

$$r = \frac{ep}{1 - e\sin\theta}. \tag{63}$$

EXAMPLE 1. Sketch the conic whose equation is

$$r = \frac{6}{3 + 2\cos\theta}.$$

Dividing the numerator and denominator by 3, we have $r = \dfrac{2}{1 + \frac{2}{3}\cos\theta}$, which is the equation of an ellipse since $e = \frac{2}{3}$. The polar axis is the principal axis. To sketch the curve all we need do is to find the intercepts. These are given below in the table, and Fig. 110 shows the sketch.

θ	r
0°	1.2
90°	2.0
180°	6.0
270°	2.0

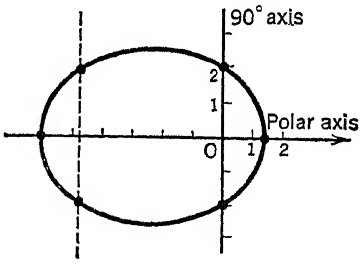


FIG. 110

Since one focus is at the pole, the other is at (4.8,180°). This allows us to find two additional points on the curve because of symmetry.

EXAMPLE 2. Sketch the conic whose equation is

$$r = \frac{4}{1 - \sin\theta}.$$

Since $e = 1$ and the equation is in the form of (63), we know that the curve is a parabola with the 90° axis as the

principal axis. The sketch of the curve and table showing its intercepts are given below. The numerator of the fraction in the equation is 4, hence $ep = 4$; and since $e = 1$, we find $p = 4$. Therefore the distance from the focus to the directrix is 4 units. On the figure the line AD is the directrix.

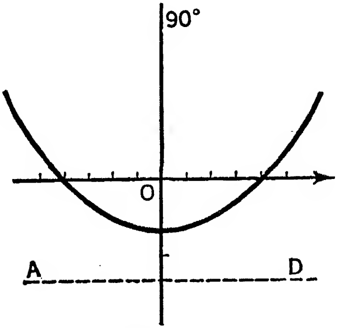


FIG. 111

θ	r
0°	4
90°	∞
180°	4
270°	2

EXERCISES

Transform the following equations into rectangular coordinates.

1. $r = 8 \sec \theta$.
2. $\tan \theta = \frac{4}{5}$.
3. $\theta = \frac{\pi}{6}$.
4. $r \sin \theta + 12 = 0$.
5. $r = 3 \cos \theta$.
6. $r = 5 \sin \theta \tan \theta$.
7. $r = 5 - 8 \sin \theta$.
8. $r = 5 \sin 2\theta$.
9. $r(5 - \cos \theta) = 10$.
10. $r(1 + \sin \theta) = 2$.

Transform the following equations into polar coordinates.

11. $x + y = 0$.
12. $x = 5$.
13. $x^2 + y^2 = 16$.
14. $x^2 - 4x + y^2 = 0$.
15. $x^2 + y^2 - 4x - 4y = 0$.
16. $4x^2 + 9y^2 = 36$.
17. $y^2 = 8x$.
18. $x^2 - y^2 = 16$.
19. $xy = 12$.
20. $5x^2 - 12y^2 = 60$.

By finding the value of e, determine the conic represented by each of the following equations and sketch the curve.

21. $r = \dfrac{4}{1 - \sin\theta}$.

22. $r = \dfrac{5}{2 + 3\sin\theta}$.

23. $r = \dfrac{8}{4 - 5\cos\theta}$.

24. $r = \dfrac{[illegible]}{3 + 2\sin\theta}$.

25. $r(4\cos\theta - 2) = -18$.

26. $(3 - 6\sin\theta)r = 12$.

27. $(1 - \cos\theta)r = 10$.

28. $(2 + 3\sin\theta)r = 4$.

29. $(3\sin\theta + 2)r = 5$.

30. $(3\cos\theta - 4)r = -12$.

31. $r = \dfrac{-3}{\cos\theta - 4}$.

32. $r = \dfrac{2}{1 + \sin\theta}$.

33. $r = \dfrac{20}{3 - \cos\theta}$.

34. $r = \dfrac{20}{5 + 15\sin\theta}$.

35. $r(4 - 8\sin\theta) = -20$.

36. $r(1 + \cos\theta) + 12 = 0$.

37. $r(1 + 3\cos\theta) + 5 = 0$.

38. $r(3\sin\theta - 2) = 8$.

39. $r(2\sin\theta + 3) = -4$.

40. $r(10 - 10\cos\theta) = -4$.

CHAPTER VII

ELEMENTS OF SOLID ANALYTIC GEOMETRY

77. Introduction. The writings of Descartes show that he thought it feasible to apply the analytical method to the geometry of three dimensions, but there is no evidence that he actually worked in this field. It was not long, however, before others, Van Schooten (1657), Parent (1700) and later eighteenth century mathematicians, extended the method, thereby developing the body of material which is called solid analytic geometry, or space [1] geometry. The remaining pages of our text will be taken up with a brief introduction to this subject, the order of development following very closely that used in plane analytic geometry.

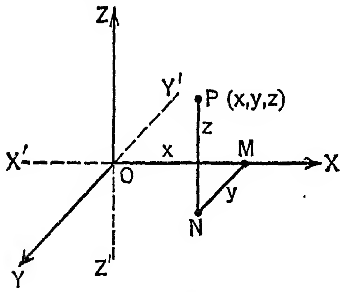


FIG. 112

THE POINT

78. Rectangular Coordinates. Through a fixed point O in space (Fig. 112) draw three mutually perpendicular lines, $X'OX$, $Y'OY$ and $Z'OZ$, and call them the x, y and z axes, respectively. Call their point of intersection O, the **origin,** and indicate positive direction along an axis by means of an arrowhead. Each pair of axes determines a plane, the xy**-plane,** the yz**-plane** or the zx**-plane** according to the axes it contains. The three planes, which divide the space into *octants,* are called **coor-**

[1] Whenever the word *space* is used reference is to the familiar space of three dimensions in which we live.

dinate planes and are used in determining the position of a point in space. Thus, to every point P there correspond three directed distances x, y and z, measured along lines perpendicular to the yz, the zx and the xy planes, respectively. Conversely, any three directed numbers x, y and z, measured in the manner indicated above, will determine the position of a point P.

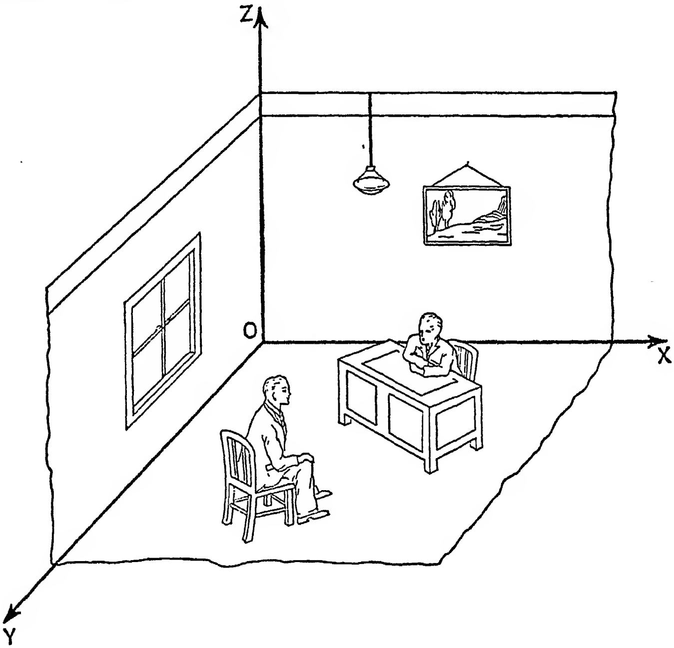


FIG. 113

These numbers, written (x,y,z), are called the **coordinates** of the point P.

When plotting points (Fig. 112), it is customary to measure the directed distance $OM = x$ along the x-axis; the directed distance $MN = y$ parallel to the y-axis; and finally, the directed distance $NP = z$ parallel to the z-axis.

To visualize the system more clearly, consider a rectangular

room (Fig. 113). Let the point in the corner where two side walls and the floor come together be the origin; the line of intersection of the right-hand wall and the floor, the x-axis; the intersection of the left-hand wall and the floor, the y-axis; and the intersection of the two walls, the z-axis. Then, the floor is the xy-plane; the right-hand wall is the zx-plane and the left-hand wall is the yz-plane. By imagining the walls and the floor extending through the axes, we conceive of eight such rooms coming together at O, four on the same level as ours and four on the floor below us. If positive directions are as indicated on the figure, we are sitting in the first octant. Going around the z-axis in a counter-clockwise direction, we come to octants II, III, and IV on the same level as octant I. The room directly below us is octant V, the one below octant II is octant VI, etc. To locate the position of an object in the room, such as an electric fixture hanging from the ceiling, we measure its perpendicular distances from the two walls and the floor.

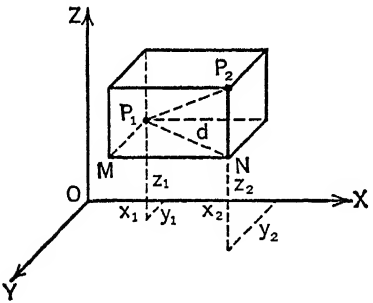


FIG. 114

79. The Distance between Two Points. To find the distance between two points in space, such as $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ in Fig. 114, we make use of the Pythagorean theorem. Thus, in the right triangle P_1MN, we see that

$$MN = x_2 - x_1, \qquad P_1M = y_2 - y_1,$$

and therefore

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

But d and the distance $NP_2 = z_2 - z_1$ are sides of the right triangle P_1NP_2. Hence the distance between the points P_1

and P_2 is expressed in terms of the coordinates of the two points by the equation

$$P_1P_2 = \sqrt{d^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad (64)$$

EXAMPLE. Plot the points $(-5,2,-4)$, $(3,2,5)$ and $(3,2,-4)$, and use the distance formula to show that the lines joining them form a right triangle.

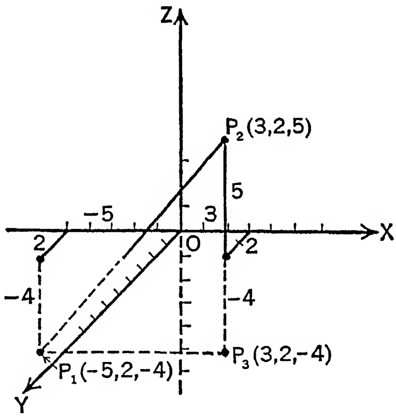


FIG. 115

In Fig. 115, let P_1, P_2 and P_3 represent the points. Then

$$P_1P_2 = \sqrt{(3+5)^2 + (2-2)^2 + (5+4)^2} = \sqrt{145},$$

$$P_2P_3 = \sqrt{(3-3)^2 + (2-2)^2 + (5+4)^2} = 9,$$

and $P_1P_3 = \sqrt{(3+5)^2 + (2-2)^2 + (-4+4)^2} = 8.$

Hence $(P_1P_2)^2 = (P_2P_3)^2 + (P_1P_3)^2$, and the triangle $P_1P_2P_3$ has a right angle at P_3.

EXERCISES

1. Plot the following points: (1,0,0), (2,0,3), (−1,−2,0), (3,5,−2), (−6,−4,2), (−3,−2,−5) and (5,−2,4) on the same set of axes.

2. What are the coordinates of a point on the x-axis? the y-axis? the z-axis?

3. What coordinate is zero for any point in the xy-plane? the zx-plane? the yz-plane?

4. What are the coordinates of the origin?

5. Find the distance from the origin to the point (2,3,−4).

6. Find the distance from the origin to any point (x_1,y_1,z_1) in space.

7. What line is determined by $x = 0$, $y = 0$? by $x = 0$, $z = 0$? by $y = 0$, $z = 0$?

8. A point in the xy-plane moves so that it is always 2 units from the point (2,5,0). What is its locus?

9. Find the distance between the points (−2,−3,4) and (5,−1,2).

10. Show that the lines joining the points (−4,−2,−3), (2,−3,4) and (3,4,−2) form an equilateral triangle.

11. Show that the lines joining the points (1,5,0), (8,5,0) and (8,−2,0) form an isosceles right triangle.

12. Find the locus of a point for which $y = 3$; for which $z = -2$; for which $y = 3$ and $z = -2$ at the same time.

13. Find the locus of a point which moves so that it is always equidistant from the points (3,−5,2) and (−4,1,−6).

14. Find the locus of a point which moves so that it is always 5 units from the origin.

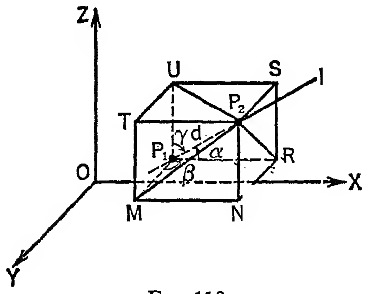


FIG. 116

80. Direction Cosines of a Line. The angles which a directed line makes with the positive directions of the coordinate axes are called the **direction angles** of the line, and the cosines of these angles are called the **direction cosines** of the line. Thus (Fig. 116), if P_1R,

P_1M and P_1U have the same directions as the x, y and z axes, respectively, and if a directed line l, passing through $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, makes angles α, β, γ with these lines, as shown in the figure, then these angles are the direction angles of l and their cosines are the direction cosines of l.[1]

Triangles P_1RP_2, P_1MP_2 and P_1UP_2 are right triangles, and, therefore, we may write

$$\cos\alpha \quad \frac{P_1R}{P_1P_2} = \frac{x_2 - x_1}{P_1P_2},$$

$$\cos\,, \quad \frac{P_1M}{P_1P_2} = \frac{RN}{P_1P_2} = \frac{y_2 - y_1}{P_1P_2}$$

$$\cos\gamma = \frac{P_1U}{} = \frac{NP_2}{P_1P_2} = \frac{z_2 - z_1}{P_1P_2}$$

or

$$x_2 - x_1 = d\cos\alpha,\ y_2 - y_1 = d\cos\beta,\ z_2 - z_1 = d\cos\gamma, \tag{65}$$

where $d = P_1P_2$. By squaring these expressions and taking their sum, we have

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = d^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

But the left member of this equation is d^2 by (64); hence we find

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \tag{66}$$

an important relation, which says that *the sum of the squares of the direction cosines of a line is equal to unity.*

If P_1 of Fig. 116 is at the origin and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from this point to any point $P(x,y,z)$, equations (65) become

$$x = r\cos\alpha, \qquad y = r\cos\beta, \qquad z = r\cos\gamma. \tag{67}$$

[1] α, β and γ are always positive and $\leqq 180°$. Since they correspond to one direction along l, the angles $180° - \alpha$, $180° - \beta$ and $180° - \gamma$ will correspond to the opposite direction.

If the coordinates of a point P are (a,b,c) as in Fig. 117, the direction cosines of the line OP are, by (67),

$$\cos\alpha = \frac{a}{r}, \qquad \cos\beta = \frac{b}{r}, \qquad \cos\gamma = \frac{c}{r}, \tag{68}$$

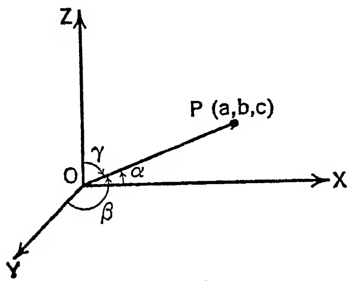


FIG. 117

where $r = \pm\sqrt{a^2 + b^2 + c^2}$. The sign of the radical depends upon the direction of the line; if it is plus when we go from O to P, then it is minus when we take the opposite direction P to O.

By solving equations (68) for r and equating like values, we find

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}. \tag{69}$$

This relation is very useful, because it tells us that the direction cosines of a line OP, or of a line parallel to OP, are proportional to the three numbers a, b and c. Such numbers are called the **direction numbers** of a line and may be used in place of the direction cosines.

EXAMPLE 1. Find the direction cosines of a line which extends from $(-6,1,-3)$ to $(2,-3,5)$.

The distance between the points is

$$r = \sqrt{(2+6)^2 + (-3-1)^2 + (5+3)^2} = 12,$$

and the direction cosines are, therefore,

$$\cos\alpha = \frac{2+6}{12} = \frac{2}{3},$$

$$\cos\beta \quad \frac{-3-1}{12} = -\frac{1}{3},$$

$$\cos\gamma \quad \frac{5+3}{12} \quad \frac{2}{3}.$$

EXAMPLE 2. A line directed upward has direction numbers , $\frac{1}{3}$ and $\frac{1}{5}$. What are the direction cosines of the line?

We have $r = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{25}} = \frac{19}{30}$; hence, by (68),

$$\cos\alpha = \qquad = \tfrac{15}{19},\ \cos\beta = \qquad = \tfrac{10}{19},\ \cos\gamma = \tfrac{1}{5}\cdot\tfrac{30}{19} = \tfrac{6}{19}$$

81. The Angle between Two Lines. The angle between two directed lines in space may be defined as the angle between two lines which pass through the origin with the same directions as the given lines. To find an expression for such an angle, which we shall call θ, let us proceed as follows. In Fig. 118, call the two lines drawn through O parallel to the given lines, l and m, and let the direction angles of l be α_1, β_1, γ_1, while those of m are α_2, β_2, γ_2. Then, by means of the Cosine Law, we may write

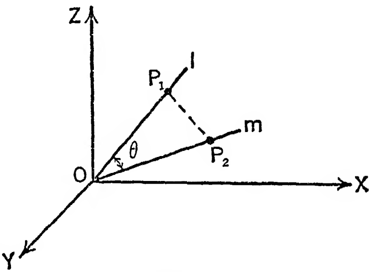


FIG. 118

$$\cos\theta \quad \frac{\ \ + (OP_2)^2 - (P_1P_2)^2}{2(OP_1)(OP_2)}.$$

But $(OP_1)^2 = x_1^2 + y_1^2$ $\qquad$ $(OP_2)^2 = x_2^2 + y_2^2 + z_2^2$, and $(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)$ $\qquad$ $- z_1)^2$, by equation (64) if the coordinates of P_1 and P_2 are (x_1,y_1,z_1) and (x_2,y_2,z_2), respectively. Hence

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y \quad + z_2^2}} = \frac{x_1x_2}{r_1r_2} + \frac{y_1y_2}{r_1r_2} + \frac{z_1z_2}{r_1r_2}.$$

where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$. But $\cos\alpha_1 = \frac{x_1}{r_1}$, $\cos\beta_1 = \frac{y_1}{r_1}$, $\cos\gamma_1 = \frac{z_1}{r_1}$, and $\cos\alpha_2 = \frac{x_2}{r_2}$, $\cos\beta_2 = \frac{y_2}{r_2}$, $\cos\gamma_2 = \frac{z_2}{r_2}$; therefore, by substituting the values of $\frac{x_1}{r_1}$, etc., we obtain

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2, \tag{70}$$

which expresses θ in terms of the direction cosines of the lines.

If the lines are perpendicular, $\cos\theta = 0$, and we have

$$\boldsymbol{\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0.} \tag{71}$$

If the lines are parallel, $\cos\theta = 1$ or -1 according to whether the lines have the same or opposite directions, and we may write

$$\boldsymbol{\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = \pm 1.} \tag{72}$$

Example. Find the angle which the line through $(-2,5,-2)$ and $(0,1,3)$ makes with the line joining $(1,2,-3)$ to $(1,0,1)$.

The direction cosines of the first line are

$$\cos\alpha_1 = \frac{2}{3\sqrt{5}},\ \cos\beta_1 = \frac{-4}{3\sqrt{5}},\ \cos\gamma_1 = \frac{5}{3\sqrt{5}};$$

those of the second are $\cos\alpha_2 = 0$, $\cos\beta_2 = \frac{-1}{\sqrt{5}}$, $\cos\gamma_2 = \frac{2}{\sqrt{5}}$.

Therefore, $$\cos\theta = \frac{2}{3\sqrt{5}}\cdot 0 + \frac{4}{3\sqrt{5}}\cdot\frac{1}{\sqrt{5}} + \frac{5}{3\sqrt{5}}\cdot\frac{2}{\sqrt{5}} = \frac{14}{15},$$

and $$\theta = \text{arc cosine } \frac{14}{15}.$$

EXERCISES

Find the direction cosines of the lines joining the following pairs of points:

1. $(0,0,0)$ to $(1,5,3)$.

2. $(1,2,3)$ to $(4,5,6)$.

3. $(-1,2,-3)$ to $(3,-4,2)$.

4. $(-1,-2,3)$ to $(-2,4,8)$.

5. $(-1,2,-5)$ to $(-4,-3,6)$.

6. $(5,8,-2)$ to $(7,-3,4)$.

Find the direction cosines of the lines whose direction numbers are the following.

7. $2, -5$ and 4.

8. $1, \frac{1}{3}$ and 2.

9. $-2, -5$ and 6.

10. $6, -3$ and -4.

11. $\frac{1}{2}, \frac{2}{3}$ and $-\frac{3}{5}$.

12. $\frac{4}{7}, -\frac{2}{5}$ and $\frac{3}{8}$.

13. Prove that it is impossible for $\frac{1}{3}, \frac{2}{5}$ and $-\frac{1}{2}$ to be the direction cosines of a line.

14. If $\frac{2}{3}$ and $\frac{1}{2}$ are two of the direction cosines of a line, find the remaining one.

15. A line makes equal angles with the coordinate axes; find its direction cosines.

16. Find the angle between the line joining $(-2,3,5)$ to $(1,-4,2)$ and the line joining $(-1,-3,-2)$ to $(-1,4,5)$.

17. Find the direction cosines of each of the coordinate axes.

18. The direction numbers of two lines are $3, 6, 2$ and $-6, 2, 3$. Show that the lines are perpendicular.

19. Show that the points $(\frac{5}{2},2,-\frac{3}{2})$, $(1,\frac{5}{2},-\frac{1}{2})$ and $(-\frac{7}{2},4,\frac{5}{2})$ are collinear.

20. Show that the points $(2,5,\frac{1}{2})$, $(\frac{1}{2},2,-\frac{1}{2})$ and $(3,\frac{7}{2},\frac{7}{2})$ are vertices of a right triangle.

21. Show that the line through $(-1,2,-\frac{7}{2})$ and $(\frac{3}{2},1,-2)$ is parallel to the line joining $(2,1,3)$ to $(-\frac{1}{2},2,\frac{3}{2})$.

22. Show that the points $(1,\frac{3}{4},\frac{5}{4})$, $(-\frac{1}{2},\frac{1}{4},\frac{1}{2})$, $(\frac{1}{4},\frac{9}{4},\frac{7}{4})$ and $(-\frac{5}{4},\frac{7}{4},1)$ are the vertices of a square.

23. Show that the points $(6,10,6)$, $(6,-8,4)$, $(-2,-2,8)$ and $(14,4,2)$ are the vertices of a rectangle.

THE PLANE

82. The Normal Form. Let a plane intersect the coordinate axes in the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ as shown in Fig. 119. The directed lengths a, b and c are called the x, y and z **intercepts,** respectively, and the lines l_1, l_2 and l_3 where the plane intersects the coordinate planes are called the **traces** of the cutting plane.

Draw a line, with direction angles α, β and γ, through O

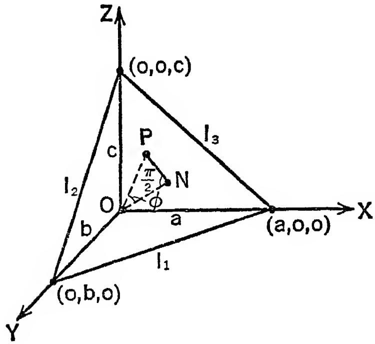


FIG. 119

perpendicular to the plane and intersecting it at N. This line is called the **normal** to the plane and its length ON is usually represented by p, a positive number when measured from the origin to the plane. Let $P(x,y,z)$ be any point on the plane. Draw the lines OP and NP and let the direction angles of OP be represented by α_1, β_1 and γ_1. Then, if ϕ is the angle between ON and OP, we have, by equation (70),

$$\cos\phi = \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1.$$

But $\cos\phi = \frac{p}{OP}$, since triangle ONP is a right triangle. Also,

$\cos\alpha_1 = \frac{x}{OP}$, $\cos\beta_1 = \frac{y}{OP}$ and $\cos\gamma_1 = \frac{z}{OP}$. Substituting these values, we have

$$\frac{p}{OP} = \frac{x}{OP}\cos\alpha + \frac{y}{OP}\cos\beta + \frac{z}{OP}\cos\gamma,$$

or

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0, \qquad (73)$$

which is known as the *normal form* of the equation of the plane.

EXAMPLE. If the perpendicular distance of a plane from the origin is 3 units and if the direction cosines of a line along this perpendicular are $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{2}{3}$, find the equation of the plane.

We have $p = 3$, $\cos\alpha = \frac{1}{3}$, $\cos\beta = \frac{2}{3}$ and $\cos\gamma = \frac{2}{3}$. Hence, by equation (73),

$$\tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z - 3 = 0, \qquad \text{or} \qquad x + 2y + 2z - 9 = 0$$

is the required equation.

83. The General Equation. We shall now show that the general equation of the first degree in three variables,

$$Ax + By + Cz + D = 0, \tag{74}$$

where the coefficients A, B and C are real and at least one of them is different from zero, represents a plane.

To do this, we divide each term of the equation by $\pm\sqrt{A^2 + B^2 + C^2}$ and compare the resulting equation with the normal form found above. Following the indicated steps, we have

$$\frac{Ax}{\pm\sqrt{A^2 + B^2 + C^2}} + \frac{By}{\pm\sqrt{A^2 + B^2 + C^2}} + \frac{Cz}{\pm\sqrt{A^2 + B^2 + C^2}} + \frac{D}{\pm\sqrt{A^2 + B^2 + C^2}} = 0$$

which is to be compared with

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

In this way, we find that (74) is the equation of a plane perpendicular to a line through the origin with direction cosines $\dfrac{A}{\pm\sqrt{A^2 + B^2 + C^2}}$, $\dfrac{B}{\pm\sqrt{A^2 + B^2 + C^2}}$ and $\dfrac{C}{\pm\sqrt{A^2 + B^2 + C^2}}$ and having a normal distance $p = \dfrac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}$. The sign of the radical $\sqrt{A^2 + B^2 + C^2}$ is so chosen that p is positive, that is, the sign of p is opposite to that of D.

Example. Reduce the equation $2x - 5y + 4z - 12 = 0$ to the normal form.

Divide each term by $\sqrt{4 + 25 + 16} = 3\sqrt{5}$ and obtain the desired equation,

$$\frac{2}{3\sqrt{5}}x - \frac{5}{3\sqrt{5}}y + \frac{4}{3\sqrt{5}}z - \frac{4}{\sqrt{5}} = 0.$$

From this form of the equation, we see that the perpendicular distance of the plane from the origin is $\frac{4}{\sqrt{5}}$ units and that the direction cosines of the normal are $\frac{2}{3\sqrt{5}}$, $-\frac{5}{3\sqrt{5}}$ and $\frac{4}{3\sqrt{5}}$.

84. A Plane Determined by Three Conditions. Since $Ax + By + Cz + D = 0$ is the general equation of a plane, the equation of a particular plane may be determined if the conditions on the plane are such that three of the coefficients A, B, C and D can be found in terms of the fourth.

Example. Find the equation of the plane passing through the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

Substituting the coordinates in the equation

$$Ax + By + Cz + D = 0,$$

we have

$$aA + D = 0, \qquad bB + D = 0 \qquad \text{and} \qquad cC + D = 0.$$

Hence, $A = -\frac{D}{a}$, $B = -\frac{D}{b}$ and $C = -\frac{D}{c}$, and our equation becomes

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0,$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{75}$$

This form is called the *intercept form* of the equation of the plane since it has intercepts a, b and c on the x, y and z axes, respectively. (See Fig. 119.)

EXERCISES

1. Find the equation of the plane for which $\alpha = 60°$, $\beta = 60°$, $\gamma = 45°$ and $p = 5$.

2. The direction numbers of the normal to a plane are 3, $-\frac{3}{2}$ and

-1, and the plane is 3 units from the origin measured along the normal. Find the equation of the plane.

3. The normal to a plane pierces it at the point $(2,1,-2)$. Find the equation of the plane.

4. The direction cosines of the normal to a plane are $\frac{1}{3}$, $-\frac{2}{3}$ and $\frac{2}{3}$, and the plane is at a distance of 8 units from the origin measured along the normal. Find the equation of the plane.

5. Reduce the equation $5x + 2y - 3z + 4 = 0$ to the normal form.

6. Reduce the equation $4x - 7y + 2z - 8 = 0$ to the normal form and find the perpendicular distance from the origin to the plane.

7. Sketch the plane $x + y - 2 = 0$.

8. Find the intercepts of the plane $2x - 5y + 4z - 10 = 0$ on the coordinate axes.

9. Find the equations of the traces of the plane in Exercise 8.

10. Find the general equation of the plane perpendicular to the xy-plane; perpendicular to the yz-plane; perpendicular to the zx-plane.

11. Find the general equation of the plane perpendicular to the x-axis; perpendicular to the y-axis; perpendicular to the z-axis.

12. Assuming that the *angle θ between two planes*

$$A_1x + B_1y + C_1z + D_1 = 0 \qquad \text{and} \qquad A_2x + B_2y + C_2z + D_2 = 0$$

is the same as the angle between the normals to the planes, show that

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

and also that the **condition that the two planes be perpendicular** is

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

13. Since the direction cosines of the normals to two parallel planes have the same values, except possibly in sign, show that the **condition that the two planes** of Exercise 12 **be parallel** is

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

14. Find the angle between the planes $4x - 2y + 2z - 9 = 0$ and $2x + 2y + 4z - 13 = 0$.

15. Show that the plane $2x + 3y - 4z + 5 = 0$ is parallel to the plane $4x + 6y - 8z + 19 = 0$ and is perpendicular to the plane $x + 2y + 2z - 14 = 0$.

16. Find the equation of the plane which passes through the points $(3,2,-3)$, $(-1,3,5)$ and $(5,4,-1)$.

17. Find the equation of the plane through the point $(-1,3,5)$ perpendicular to the line joining $(2,4,-3)$ to $(4,1,6)$.

18. Find the equation of the plane through the points $(2,4,6)$ and $(6,4,-2)$, and perpendicular to the plane $3x + 2y + 6z - 4 = 0$.

19. Find the coordinates of the point where the three planes, $x + y - z + 4 = 0$, $2x + 3y + 7z - 5 = 0$ and $3x - 4y + 2z = 8$ intersect.

20. Derive the condition that any three planes,

$$A_1x + B_1y + C_1z + D_1 = 0, \qquad A_2x + B_2y + C_2z + D_2 = 0$$

and $A_3x + B_3y + C_3z + D_3 = 0$, intersect in a point.

21. Find the equation of the plane through the point $(2,\frac{1}{2},0)$ perpendicular to the planes $x - y + z - 3 = 0$ and $2x + y - 3z - 8 = 0$.

22. Find the equation of the plane through the points $(8,2,4)$ and $(2,6,-2)$, perpendicular to the yz-plane.

THE LINE

85. Symmetric Equations of a Line. Consider a line segment through the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ as shown in Fig. 120. We have seen (Art. 80) that

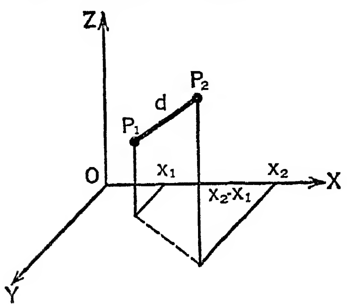


FIG. 120

$$x_2 - x_1 = d \cos \alpha,$$
$$y_2 - y_1 = d \cos \beta,$$
$$z_2 - z_1 = d \cos \gamma,$$

where $d = P_1P_2$, and α, β, γ are the direction angles of the line through P_1 and P_2. If we solve each of these equations for d and equate values, we get

$$\frac{x_2 - x_1}{\cos \alpha} = \frac{y_2 - y_1}{\cos \beta} = \frac{z_2 - z_1}{\cos \gamma}.$$

This relation states that the projections of a line segment on the coordinate axes are proportional to the direction cosines of the line. When the two points P_1 and P_2 are given, the direction cosines of the line joining them can be obtained. Likewise, if only one point (x_1,y_1,z_1) and the direction cosines are given, a line is determined. Therefore, if $P_2(x_2,y_2,z_2)$ is taken as the arbitrary point and written $P(x,y,z)$, we may write

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma} \tag{76}$$

as the equation of a straight line through the point (x_1,y_1,z_1) whose direction cosines are known.

By replacing the direction cosines by direction numbers, we have

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \tag{77}$$

which is called *the symmetric form of the equation of a straight line.*

These equations may be written as the equations of three planes, but they are not independent since any two may be combined to give the equation of the third plane. Since the coordinates of every point on the line must satisfy each of these three equations, each plane contains the line and is perpendicular to one of the coordinate axes. They are called the *projecting planes* of the line.

EXAMPLE. Find the equation of the line which passes through the point (1,3,5) with $\alpha = 30°$, $\beta = 45°$, and $\gamma = 60°$.

Using equation (76), we obtain

$$\frac{x - 1}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{\sqrt{2}}{2}} = \frac{z - 5}{\frac{1}{2}},$$

which may be simplified and expressed in the form

$$\frac{x-1}{\sqrt{3}} = \frac{y-3}{\sqrt{2}} = \frac{z-5}{1}.$$

86. The Two-Point Form of the Line. If a straight line is determined by the two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ we may write the equation of the line through these points by using equation (77). Thus, $\dfrac{x_2-x_1}{a} = \dfrac{y_2-y_1}{b} = \dfrac{z_2-z_1}{c}$ and by eliminating a, b, c between (77) and these equations, we may write

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}. \tag{78}$$

Example. Find the equation of the line through the points (3,4,5) and (5,−2,3).

Substituting these coordinates in the above form, we have

$$\frac{x-3}{2} = \frac{y-4}{-6} = \frac{z-5}{-2} \quad \text{or} \quad \frac{x-5}{-2} = \frac{y+2}{6} = \frac{z-3}{2}$$

depending upon which point we select as P_1.

87. General Equations of a Line. A straight line is determined when two planes intersect in space. If

$$A_1x + B_1y + C_1z + D_1 = 0$$

and

$$A_2x + B_2y + C_2z + D_2 = 0$$

are the equations of two intersecting planes, any point whose coordinates satisfy both equations is a point on the line determined by the two planes, and the coordinates of any point on the line will satisfy both equations. *The two equations of the planes may, therefore, be considered as the equation of the line.*

The general equation of a line may be reduced to the symmetric form by eliminating first one variable, say x, and obtaining a projecting plane in the other two variables; and then by eliminating a second variable, say y, and so obtaining another projecting plane in x and z. If the two resulting equations are solved for z, the symmetric form may be obtained by equating these values of z.

Example. Reduce the equation of the line

$$3x + 2y + 4z = -5, \qquad x - y + 2z = 4$$

to the symmetric form.

Let us first eliminate y and then x, thereby obtaining

$$z = \frac{3 - 5x}{8} \qquad \text{and} \qquad z = \frac{5y + 17}{2}.$$

Equating these values and rearranging the terms, we have

$$\frac{-5x + 3}{8} = \frac{5y + 17}{2} = z.$$

This may now be reduced to the symmetric form by changing the coefficient of x and y to unity, or,

$$\frac{x - \frac{3}{5}}{-\frac{8}{5}} = \frac{y + \frac{17}{5}}{\frac{2}{5}} = \frac{z - 0}{1}$$

Finally, this may be written as $\frac{x - \frac{3}{5}}{-8} = \frac{y + \frac{17}{5}}{2} = \frac{z - 0}{5}$, and in this form the direction cosines of the line are proportional to the denominators.

EXERCISES

Find the equations of the lines determined by the following pairs of points.

1. (3,2,−4) and (5,4,−6).
2. (−1,−2,−3) and (3,0,1).
3. (4,5,6) and (−2,−3,−4).
4. (5,4,3) and (0,1,−1).
5. (2,4,3) and (4,1,6).

6. Find the equations of the line which passes through the point (2,5,3) and has direction numbers 1, 3, 4.

7. Find the equations of the line passing through the point $(3,4,-5)$ which makes an angle of 120° with the x-axis and 60° with the z-axis.

8. Find the equation of the line through the origin perpendicular to the plane $5x - 3y + z - 2 = 0$.

9. Find the equation of the line which passes through the point $(1,-2,3)$ and is perpendicular to the plane $6x + 7y - 5z = 10$.

10. Find the equation of the line through the point (2,4,5) perpendicular to the plane $2x - 5y + 4z = 10$.

11. Reduce the equation of the line $x + 2y + 3z - 2 = 0$, $3x + 2y - z - 4 = 0$ to the symmetrical form, and determine the direction cosines.

12. Find the direction cosines for the line determined by

$$x + y - z + 4 = 0 \qquad \text{and} \qquad 2x + 3y + 7z - 5 = 0.$$

13. Find the direction cosines for the line determined by

$$3x - 2y + z - 1 = 0 \qquad \text{and} \qquad x + y - 2z + 2 = 0.$$

14. Find the points in which the line $\dfrac{x-2}{2} = \dfrac{y-1}{3} = \dfrac{z - \text{[illegible]}}{-2}$, meets the coordinate planes.

15. Find the points in which the line determined by

$$x + 2y - 3z - 1 = 0 \qquad \text{and} \qquad 3x - 2y + 5z - 3 = 0$$

meets the coordinate planes.

SURFACES

88. The Sphere. We may define a **sphere** as the locus of a point which moves in such a way that it is always a constant distance from a fixed point. As in the case of a circle in the plane, the fixed point is called the *center* and the fixed distance the *radius*. To derive the equation, let (h,k,l) denote the center and r, the radius. Using the distance formula, we have

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2. \tag{79}$$

This equation is satisfied only by points on the sphere and is, therefore, the equation of the sphere.

If the center is at the origin, we have $h = k = l = 0$ and equation (79) will reduce to

$$x^2 + y^2 + z^2 = r^2.$$

We may obtain a more general form of the equation of a sphere by expanding equation (79) and collecting terms as follows:

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0. \tag{80}$$

By completing the squares, this equation may be transformed into

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + \left(z + \frac{C}{2}\right)^2 = \frac{A^2 + B^2 + C^2 - 4D}{4},$$

and by comparing it with equation (79) we see that it represents a sphere with center $\left(-\frac{A}{2}, -\frac{B}{2}, -\frac{C}{2}\right)$ and radius $r = \frac{1}{2}\sqrt{A^2 + B^2 + C^2 - 4D}$. In order to have a real sphere, we must have $r > 0$, or $A^2 + B^2 + C^2 - 4D > 0$. If $r = 0$, the sphere is called a *point sphere*; and if $A^2 + B^2 + C^2 - 4D < 0$ there is no locus.

Example 1. Find the equation of the sphere of radius $\frac{2}{3}$ and center $(2, -3, 0)$.

In this case, we have $h = 2$, $k = -3$, $l = 0$, and $r = \frac{2}{3}$. Substituting these values in (79), we have

$$(x - 2)^2 + (y + 3)^2 + (z - 0)^2 = \frac{4}{9}$$

as the equation of the sphere; or, by simplifying, the equation may be written in the general form

$$9x^2 + 9y^2 + 9z^2 - 36x + 54y + 113 = 0.$$

Example 2. Find the center and radius of the sphere whose equation is

$$4x^2 + 4y^2 + 4z^2 + 16x - 24y - 8z + 31 = 0.$$

By dividing each term by 4 and then transposing the constant term, we have

$$x^2 + y^2 + z^2 + 4x - 6y - 2z = -\tfrac{31}{4}.$$

By completing the squares in x, in y, and in z, we get

$$(x + 2)^2 + (y - 3)^2 + (z - 1)^2 = \tfrac{25}{4}.$$

Therefore the center is $(-2,3,1)$ and the radius is $r = \frac{5}{2}$.

89. The Cylinder. A surface which is generated by a straight line moving parallel to itself and intersecting a fixed curve, is known as a **cylinder.** The fixed curve is called the *directrix* and the moving line, a *generator* or *element*. In the following examples we shall consider equations whose loci are cylinders.

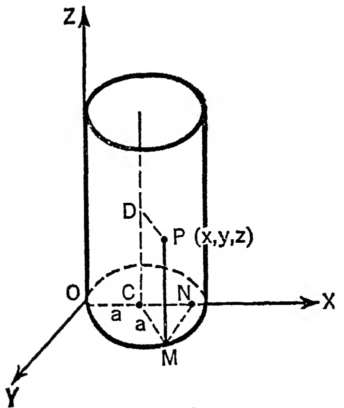


FIG. 121

EXAMPLE 1. Find the equation of a right circular cylinder of radius a whose axis intersects the x-axis and is parallel to the z-axis.

Consider the cylinder drawn in Fig. 121 with its axis CD intersecting the x-axis at point C. Let $P(x,y,z)$ be any point on the cylinder. Then, $ON = x$, $NM = y$, and $MP = z$. If DP is drawn perpendicular to CD, we have $DP = CM = a$ as the condition that the point P is on the cylinder. However, in the right triangle CNM,

$$(CM)^2 = (CN)^2 + (NM)^2.$$

Substituting $CM = a$, $CN = x - a$ and $NM = y$, we get

$$a^2 = (x - a)^2 + y^2, \qquad \text{or} \qquad x^2 + y^2 - 2ax = 0,$$

as the equation of the cylinder. The directrix of the cylinder is

the circle $x^2 + y^2 - 2ax = 0$ in the xy-plane. *It is to be noticed that the equation of the cylinder contains only two variables.*

EXAMPLE 2. Determine the nature of the surface whose equation is $y^2 = 4ax$.

In the xy-plane this represents a parabola. If $Q(x,y,0)$ represents any point of the parabola $y^2 = 4ax$, the coordinates (x,y,z) of the point P will also satisfy the equation and z may have any value. This means that any point on the line through Q parallel to the z-axis will lie on the locus. Therefore, all lines which pass through points on the parabola and are parallel to the z-axis will lie on the surface represented by the equation $y^2 = 4ax$. Since this surface is generated by a line which moves so as to be parallel to itself, it is called a **parabolic cylinder** and the parabola is its *directrix.*

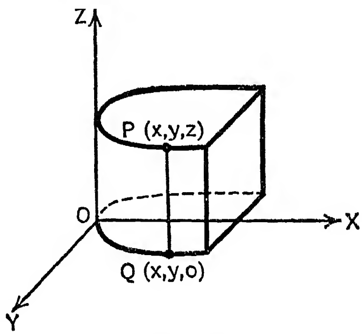


FIG. 122

The two examples above indicate that the locus of an equation, in which one variable is missing, represents a cylinder with elements parallel to the axes corresponding to that variable. The given equation represents the plane locus of the directrix.

EXERCISES

Find the equation of the following spheres.

1. Center (2,3,4), radius = 5.

2. Center $(-2,3,-8)$, radius = 4.

3. Center $(-\frac{3}{2},\frac{1}{2},1)$, radius $= \frac{5}{2}$. **4.** Center (2,0,5), radius = 3.

Determine the center and radius of the following spheres.

5. $x^2 + y^2 + z^2 + 6x - 4y + 14z + 46 = 0.$

6. $x^2 + y^2 + z^2 - 8x + 8y - 2z + 24 = 0.$

7. $4x^2 + 4y^2 + 4z^2 - 16x + 8y - 32z + 59 = 0.$

8. $2x^2 + 2y^2 + 2z^2 - 16y - 20z + 81 = 0.$

9. Find the equation of the sphere which has the line segment joining $(2,-3,-2)$ and $(4,-5,3)$ as a diameter.

10. Find the equation of the sphere with center $(2,-1,3)$ and tangent to (a) xy-plane; (b) yz-plane; and (c) xz-plane.

Sketch the following surfaces:

11. $x + 2y - 2 = 0.$

12. $y^2 = 4z.$

13. $x^2 + y^2 = 4.$

14. $x^2 - 2y^2 = 1.$

15. $x^2 + z^2 - 4z = 0.$

16. $y^2 - 9z = 0.$

90. Traces. We may define the equation of a surface as that equation which is satisfied by the coordinates of each point on the surface and by the coordinates of no other points. A plane will intersect a surface, in general, in a curve. To sketch the figure of a surface, it is helpful to find the curve of intersection of the surface with the coordinate planes, and with planes parallel to the coordinate planes. Such curves are called **traces** of the surface. We shall illustrate the method of using these traces in the following example.

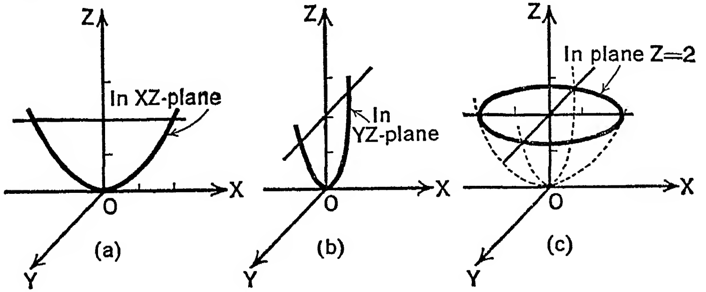


FIG. 123

EXAMPLE. The surface $4x^2 + y^2 = 8z$ is intersected by the following planes: (a) $y = 0$; (b) $x = 0$; and (c) $z = 2$. Find and draw the curve of intersection in each case.

(a) Substituting $y = 0$ in the equation of the surface, we

obtain $x^2 = 2z$, as the equation of the curve, or the trace of the surface, in the xz-plane. This parabola is sketched in (a) of Fig. 123.

(b) If $x = 0$, we obtain the parabola $y^2 = 8z$ which is the curve of intersection, or trace, in the yz-plane, and is shown in (b) of the figure.

(c) Letting $z = 2$, we have $4x^2 + y^2 = 16$ as the curve of intersection of a plane parallel to the xy-plane and 2 units above it. This ellipse is shown in part (c) of the figure. The complete surface may be sketched by showing these traces on the same reference frame, as in (c) of Fig. 123.

91. Quadric Surfaces. If the equation of a surface is of the second degree in the variables x, y, and z, the surface is called **a quadric surface.** In space, the quadric surfaces play the same role as do the conic sections in the plane. We shall consider certain typical quadric surfaces and illustrate them by figures. The same consideration of symmetry and intercepts used in drawing the locus of an equation in the plane will be useful here, as well as the idea of finding the traces, as discussed in the preceding article. The following examples will illustrate the steps necessary to draw the locus of an equation in three variables.

EXAMPLE 1. Draw the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(a) The intercepts on the x-axis are $x = \pm a$; on the y-axis, $y = \pm b$; and on the z-axis, they are imaginary.

(b) Find the traces on the coordinate planes by letting $x = 0$, $y = 0$, and $z = 0$, successively. Thus, $x = 0$ gives $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, a hyperbola in the yz-plane with transverse axis $2b$; $y = 0$ gives $\dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1$, a hyperbola in the xz-plane with transverse axis $2a$,

$z = 0$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse in the xy-plane with major axis $2a$ and minor axis $2b$.

(c) Pass a plane, $z = k$, through the surface parallel to the xy-plane. This gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2},$$

or

$$\frac{x^2}{\frac{a^2}{c^2}(c^2 + k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 + k^2)} = 1,$$

and indicates that every section parallel to the xy-plane is an ellipse.

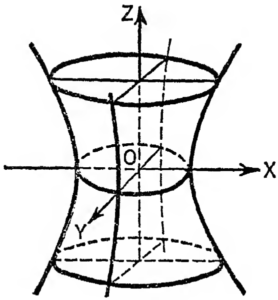


Fig. 124

This surface is illustrated in Fig. 124 and is known as a **hyperboloid of one sheet.**

Example 2. Draw the locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This surface intersects the x-axis in the points $x = \pm a$, while the y and z intercepts are imaginary.

The trace in the xz-plane, found by letting $y = 0$, is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$. The trace in the xy-plane is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. There is no trace of this surface in the yz-plane.

To find the trace in a plane parallel to the yz-plane, let $x = k$. This gives the equation

$$\frac{y^2}{b^2\left(\frac{k^2}{a^2} - 1\right)} + \frac{z^2}{c^2\left(\frac{k^2}{a^2} - 1\right)} \quad 1.$$

We notice that if $-a \leqq k \leqq a$, there is no real ellipse. If

$k > a$ or $k < -a$, the traces are real ellipses which increase in size as the plane $x = k$ moves away from the yz-plane in either direction. Consequently, there is no part of the locus between the planes $x = a$ and $x = -a$, that is, the surface is divided into two parts. It is called a **hyperboloid of two sheets** and is shown in Fig. 125.

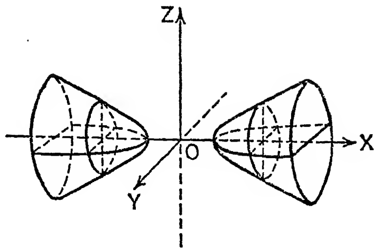


FIG. 125

EXAMPLE 3. Draw the locus of $\frac{x^2}{a^2} - \frac{y^2}{b^2} \quad cz$, where $c > 0$.

Using the method of the preceding examples, we find that this locus has zero intercepts on the x, y, and z axes. The trace of the surface in the yz-plane is the parabola $y^2 = -b^2cz$, with axis along the negative z-axis. The trace in the xz-plane is the parabola $x^2 = a^2cz$, whose axis is the positive z-axis. In the xy-plane the trace consists of two intersecting lines, $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$. The plane $z = k$ cuts the surface in the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = kc$. If $k < 0$, the transverse axis is parallel to the x-axis, and if $k < 0$, the transverse axis is parallel to the y-axis. Also, the cross section made by a plane parallel to the yz-plane is a parabola with constant latus rectum which has its vertex on the boundary parabola in the xz-plane. This surface is called a **hyperbolic paraboloid,** and is shown in Fig. 126.

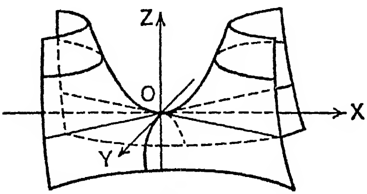


FIG. 126

EXERCISES

Sketch the following surfaces by finding traces on the coordinate planes and on planes parallel to the coordinate planes.

1. $\frac{x^2}{9} + \frac{y^2}{4} + z = 1.$
2. $\frac{x^2}{16} + \frac{y^2}{9} = 4z.$
3. $\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{4} = 1.$
4. $\frac{y^2}{9} + \frac{z^2}{16} = 6x.$
5. $x^2 - y^2 - z^2 = 1.$
6. $4x^2 - 9y^2 + z^2 + 36 = 0.$
7. $6x^2 + 3y^2 + 2z^2 = 6.$
8. $x^2 + z^2 = 9y^2.$
9. $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{4} = 1.$
10. $\frac{x^2}{36} - \frac{y^2}{9} + \frac{z^2}{36} = 1.$
11. $\frac{x^2}{25} - \frac{y^2}{9} = 4z.$
12. $12x^2 + 9y^2 - 16z = 0.$
13. $x^2 - y^2 + 4x = 0.$
14. $4x^2 + 9y^2 = 36z.$
15. $y^2 + 4z^2 = 4x^2.$
16. $4x^2 + 3y^2 = 12z.$

ANSWERS TO ODD-NUMBERED EXERCISES

Pages 5, 6

3. Rectangle. **5.** $x = 0; y = 0$. **9.** $(-6,-4)$.
15. $(5,3)$. **17.** $(4,2); (7,5)$.

Pages 10, 11

1. $8; -6$. **3.** $(-\frac{3}{2},2)$. **5.** $(2,-3); (8,1); (2,1)$.
7. $A_1 = \frac{1}{2}(c - a)(d - b); A_2 = \frac{1}{8}(c - a)(d - b); A_1 = 4A_2$
9. $\sqrt{130}; (\frac{7}{2},\frac{5}{2})$. **13.** $\sqrt{89}; 5\sqrt{5}$.
17. $(3,4)$. **21.** $(\frac{37}{2},5)$.
23. $(0,\pm 4\sqrt{3})$.

Pages 17, 18

1. a. $\frac{5}{2}$. **b.** $-\frac{6}{5}$. **c.** 2. **d.** -11. **e.** $\frac{1}{7}$. **f.** $-\frac{1}{3}$.
3. $m = 0; \alpha = 0°$. m undefined; $\alpha = 90°$.
15. $\frac{69}{13}; \frac{23}{44}; \frac{23}{7}$. **23.** $\theta = 90° + 2\alpha$. **25.** $x - y - 1 = 0$.

Pages 30, 31

1. $x - 2 = 0; y = 0$. **3.** $y + 3 = 0; x = 0$.
5. $x - 4 = 0; y = 0$. **7.** $x = 0; x - 2 = 0; y = 0$.
9. $x - 6 = 0; y \pm 2 = 0$. **11.** $3x - 4 = 0; 3y - 2 = 0$.
13. $x - 2 = 0; x - 1 = 0; y = 0$. **15.** $(3,2); (2,3)$.
17. $(\frac{9}{2},6); (\frac{1}{2},-2)$. **19.** $(\sqrt{17},\pm 2\sqrt{2}); (-\sqrt{17},\pm 2\sqrt{2})$.
21. $(\frac{1}{2}\sqrt{26},\pm\frac{1}{2}\sqrt{10}); (-\frac{1}{2}\sqrt{26},\pm\frac{1}{2}\sqrt{10})$.
23. $(\pm\frac{5}{4}\sqrt{7},\pm\frac{9}{4})$. **25.** $(\pm\sqrt{2},\pm\sqrt{3})$.
27. $(5,3); (-\frac{24}{5},-\frac{53}{15})$.

Pages 33, 34

1. $x^2 + y^2 + 4x - 6y - 3 = 0$. **3.** $y^2 = 8x$. **5.** $x = \pm 1$.
7. $3x^2 + 3y^2 + 20x + 12 = 0$. **9.** $x^2 + y^2 + 8x - 6y = 0$.
11. $y^2 - 8x + 16 = 0$.

Pages 37, 38

1. (a) $(-2,0)$, $(-6,-2)$, $(-6,-4)$; (b) $(0,2)$, $(-4,0)$, $(-4,-2)$; (c) $(2,8)$, $(-2,6)$, $(-2,4)$; (d) $(4,7)$, $(0,5)$, $(0,3)$. (e) $(-\frac{1}{2},\frac{5}{2})$,

3. $2x' - 3y' - 1 = 0$

5. $x'^2 + y'^2 = 25$.

7. $3y'^2 - 7x' = 0$.

9. $y'^2 = 4x' + 4$.

11. $x'^2 + y'^2 = 4$.

13. $4x'^2 + 4y'^2 = 89$.

15. $3x'^2 - 3x'y' - y'^2 + 1 = 0$.

17. $2x'^2 + 2y'^2 = 25$.

19. $8x'^2 - y'^2 = 24$.

21. $4x'^2 + 4y'^2 = 29$.

23. $9x'^2 - y'^2 = 9$.

25. $100x'^2 + 100y'^2 = 61$.

27. $x'^2 - 9y'^2 = 25$.

29. $4x' + 3y' = 0$; $x' + y \quad = 0$.

Pages 44, 45, 46

1. $2x - 3y + 12 = 0$.

3. $3x - y - 2 = 0$.

5. $5x - 7y - 22 = 0$.

7. $4x - y - 37 = 0$.

9. $2x + 5y - 19 = 0$.

11. $6x - 11y + 12 = 0$.

13. $23x + 15y - 2 = 0$.

15. (a) $5x + 4y - 25 = 0$; (b) $4x - 5y + 21 = 0$.

17. (a) $5x + 3y - 2 = 0$, $3x - 5y - 8 = 0$, $x + 4y - 14 = 0$; (b) $7x + 11y - 30 = 0$, $4x - y - 5 = 0$, $x - 13y + 20 = 0$; (c) $3x - 5y + 9 = 0$, $5x + 3y - 19 = 0$, $4x - y - 5 = 0$; (d) $3x - 5y + 26 = 0$, $5x + 3y - 36 = 0$, $x + 4y + 3 = 0$; (e) $45°$, $45°$, $90°$.

19. $m = \frac{3}{5}$, $b = -2$.

21. $m = -\frac{4}{3}$, $b = 6$.

23. $m = -3$, $b = 7$.

25. $m = 0$, $b = -\frac{2}{3}$.

27. (a) $k = -12$, $-\frac{1}{\text{[illegible]}}$ (b) $k = \frac{4}{3}, \frac{20}{3}$.

29. $k = \frac{1}{2}$.

31. $x + 2y = 13$.

33. $8x + 3y - 24 = 0$, $2x + 3y - 12 = 0$.

35. $2x - 3y + 5 = 0$.

37. $(\frac{1}{9}, \frac{5}{9})$, $\frac{1}{9}\sqrt{2465}$.

Page 51

1. $\sqrt{3}x + y - 12 = 0$. 3. $x + y + 4\sqrt{2} = 0$. 5. $\sqrt{3}x - y + 10 = 0$.

7. $x - \sqrt{3}y - 3 - 6\sqrt{3} = 0$, $x - 2y - 15 = 0$.

19. $-\dfrac{2\sqrt{17}}{17}$.

21. $\dfrac{6\sqrt{13}}{13}$.

23. $-\dfrac{11\sqrt{34}}{17}$.

25.

Pages 56, 57

1. $4x + 4y - 9 = 0,\ 8x - 8y + 15 = 0.$
3. $x - 13y + 126 = 0,\ 13x + y + 42 = 0.$
5. $x + 13y + 2 = 0,\ 13x - y + 26 = 0.$
7. $7x - 56y + 118 = 0, \quad 63x - 14y + 262 = 0, \quad 10x + 11y + 20 = 0;$ $\left(-\frac{2418}{637}, \frac{1040}{637}\right)$.
9. $9x + 7y - 5 = 0, \quad 2x - 14y + 23 = 0, \quad 32x - 4y + 27 = 0;$ $\left(-\frac{13}{20}, \frac{\ }{20}\right)$
11. $x + 2y = 0,\ 4x - 4y + 21 = 0,\ 2x + 7 = 0;\ (-\frac{7}{2},$
13. $(2,2),\ \frac{6}{5}\sqrt{5}.$ **15.** $13\frac{1}{2}$. **17.** 65.

Pages 61, 62

1. $y - 3 = k(x + 1),\ 2x - y + 5 = 0.$
3. $4x + 3y \pm 12 = 0.$ **5.** $2x - 5y - 14 = 0.$
7. $35x + 29y - 88 = 0.$ **9.** $3x + 2y + 35 = 0.$
11. $5x - 2y - 7 = 0.$ **13.** $x - 6y + 21 = 0.$
15. $x = 0,\ bx - cy - 2ab = 0,\ ax - cy - 2ab = 0;\ \left(0, -\frac{2ab}{c}\right)$.

Pages 66, 67

1. $x^2 + y^2 = 16.$ **3.** $x^2 + y^2 - 4x + 4y - 28 \quad 0.$
5. $x^2 + y^2 + 8x - 4y + 4 = 0.$ **7.** $x^2 + y^2 + 4x - 4y + 4 = 0.$
9. $x^2 + y^2 + 6x - 10y - 66 = 0.$ **11.** $(1,-2),\ r = 4.$
13. $(\frac{1}{2}, -1),\ r = 0.$ **15.** $(-\frac{4}{3}, -\frac{2}{3}),\ r = \frac{2}{3}\sqrt{5}.$
17. $x^2 + y^2 + x - 4 = 0.$ **19.** $7x + 4y - 1 = 0.$ **21.** $2x^2 + 2y^2 = a^2.$

Pages 70, 71

1. $x^2 + y^2 + 11x - 17y = 0.$ **3.** $x^2 + y^2 - 9x - 8y + 5 = 0.$
5. $x^2 + y^2 - 2x - 3y = 0.$ **7.** $x^2 + y^2 - 2x - 2y - 15 = 0.$
9. $x^2 + y^2 - 10x + 9y - 61 = 0.$
11. $x^2 + y^2 - 4x - 10y - 24 = 0,\ x^2 + y^2 + 6x + 8y - 28 = 0.$
13. $x^2 + y^2 - 6x - 6y + 9 = 0,\ x^2 + y^2 - 30x - 30y + 225 = 0.$
15. $25x^2 + 25y^2 - 155x - 151y - 182 = 0.$
17. $25x^2 + 25y^2 - 10x + 70y + 1 = 0.$
19. $x^2 + y^2 - 6x - 8y = 0,\ 5x^2 + 5y^2 + 48x - 14y = 0.$

Page 75

1. $4x^2 + 4y^2 - 17x - 55y + 171 = 0.$
3. $x^2 + y^2 + 28x - 46y + 75 = 0.$
5. $x^2 + y^2 - 18x + 29 = 0.$ **7.** $x^2 + y^2 - 12x - 6y + 35 = 0.$
9. $x^2 + y^2 - 42x + 24y + 5 = 0.$ **11.** $(-3,1)$, $(-1,4)$.
13. $\sqrt{13}$. **17.** $2x + 9y = 0.$

Page 79, 80

1. $(2,0)$; $(2,\pm 4)$; $x + 2 = 0.$ **3.** $(-1,0)$; $(-1,\pm 2)$; $x - 1 = 0.$
5. $(0, \frac{5}{16})$; $(\pm\frac{5}{8}, \frac{5}{16})$; $16y + 5 = 0.$
7. $(\frac{1}{2},0)$; $(\frac{1}{2},\pm 1)$; $2x + 1 = 0.$
9. $\left(0,-\dfrac{5}{12}\right)$; $\left(\pm\dfrac{5}{6}, -\dfrac{5}{12}\right)$; $12y - 5 = 0.$ **11.** $3y^2 = x.$
13. $4y^2 + 25x = 0.$ **15.** $y^2 = 9x.$
17. $2y^2 = -25x.$ **19.** $3y^2 + 8x = 0.$ **21.** $x^2 = 3y.$
23. $5x^2 + 9y = 0.$ **25.** $5x^2 + 9y = 0.$ **27.** $4x^2 + 9y = 0.$
29. $x^2 = 8y.$ **31.** $y^2 + 12x - 9 = 0.$ **33.** $9y^2 + 48x + 90y + 353 = 0.$

Pages 83, 84, and 85

(Answers to Exercises 7–13 are given in the order of the questions.)

1. $x^2 - 12x - 12y + 96 = 0.$ **3.** $y^2 + 4y - 16x + 4 = 0.$
5. $y^2 - 8x - 16 = 0.$
7. $(2,-4)$; $(0,-4)$; $(0,0)$, $(0,-8)$; $y + 4 = 0$; $x - 4 = 0.$
9. $(4,3)$; $(1,3)$; $(1,-3)$, $(1,9)$; $y - 3 = 0$; $x - 7 = 0.$
11. $(-\frac{5}{2},-1)$; $(-\frac{5}{2},-\frac{15}{16})$; $(-\frac{21}{8},-\frac{15}{16})$, $(-\frac{19}{8},-\frac{15}{16})$; $2x + 5 = 0$; $16y + 17 = 0.$
13. $(\frac{71}{84},-\frac{5}{6})$; $(\frac{10}{7},-\frac{5}{6})$; $(\frac{10}{7},\frac{1}{3})$, $(\frac{10}{7},-2)$; $6y + 5 = 0$; $42x - 11 = 0.$
15. $y^2 - 8y - 32x + 112 = 0.$ **17.** $2y^2 - 9x - 8y + 17 = 0.$
19. $y^2 + 2y - 5x - 9 = 0$; $y^2 + 2y + 5x + 11 = 0.$
21. $x^2 - x - y - 2 = 0.$ **23.** $1\frac{7}{8}$ seconds later.
25. $x^2 + y^2 - 15y = 0.$

Pages 90, 91

(Answers to Exercises 1–9 are given in the order of the questions.)

1. 2; $\sqrt{2}$; $(\pm\sqrt{2},0)$; $(\pm 2,0)$; 2; $\frac{1}{2}\sqrt{2}$.
3. $2\sqrt{3}$; 2; $(0,\pm 2\sqrt{2})$; $(0,\pm 2\sqrt{3})$; $\frac{4}{3}\sqrt{3}$; $\frac{1}{3}\sqrt{6}$.
5. $2\sqrt{2}$; 2; $(0,\pm 2)$; $(0,\pm 2\sqrt{2})$; $2\sqrt{2}$; $\frac{1}{2}\sqrt{2}$.
7. 6; 4; $(\pm 2\sqrt{5},0)$; $(\pm 6,0)$; $\frac{16}{3}$; $\frac{1}{3}\sqrt{5}$.

9. $6; 3; (\pm 3\sqrt{3},0); (\pm 6,0); 3; \frac{1}{2}\sqrt{3}.$
11. $25x^2 + 16y^2 = 400.$
13. $9x^2 + 4y^2 = 36.$
15. $5x^2 + 9y^2 = 144.$
17. $7x^2 + 16y^2 = 448.$
19. $5x^2 + 12y^2 = 128.$
21. $36x^2 + 27y^2 = 972.$
23. $\frac{1}{4}\sqrt{7}.$
25. $a^2x^2 + a^2y^2 = a^4 - a^2b^2 + b^4.$

Pages 95, 96

(Answers to Exercises 1–9 are given in the order of the questions.)

1. $(2,0); 3; 1; (-1,0), (5,0); (2 \pm 2\sqrt{2},0); \frac{2}{3}; \frac{2}{3}\sqrt{2}.$
3. $(\frac{1}{5},\frac{2}{3}); 5; 3; (\frac{1}{5},-4\frac{1}{3}), (\frac{1}{5},5\frac{2}{3}); (\frac{1}{5},4\frac{2}{3}); (\frac{1}{5},-3\frac{1}{3}); \frac{18}{5}; \frac{4}{5}.$
5. $(2,-3); 4; 2\sqrt{3}; (-2,-3), (6,-3); (0,-3), (4,-3); 6; \frac{1}{2}.$
7. $(1,-1); 5\sqrt{2}; 2\sqrt{2}; (1 \pm 5\sqrt{2},-1); (1 \pm \sqrt{42},-1); \frac{8}{5}\sqrt{2}; \frac{1}{5}\sqrt{21}$
9. $(-\frac{1}{2},1); \frac{1}{2}\sqrt{89}; \frac{1}{4}\sqrt{178}; (-\frac{1}{2} \pm \frac{1}{2}\sqrt{89},1); (-\frac{1}{2} \pm \frac{1}{4}\sqrt{178},1)$ $\frac{1}{2}\sqrt{89}; \frac{1}{2}\sqrt{2}.$
11. $9x^2 + 5y^2 + 54x - 20y + 21 = 0.$
13. $25x^2 + 16y^2 - 300x + 96y - 556 = 0.$
15. $x^2 + 4y^2 - 2x - 16y + 13 = 0.$
17. $25x^2 + 16y^2 - 50x - 32y - 359 = 0.$
19. $7x^2 + 16y^2 - 64y - 48 = 0.$
21. $a^2x^2 + b^2y^2 = a^2b^2.$

Pages 102, 103

(Answers to Exercises 1–9 are given in the order of the questions.)

1. $4; 3; 5; \frac{5}{4}; (\pm 5,0); (\pm 4,0); (\pm 5,\pm\frac{9}{4}); \frac{9}{2}; 3x \pm 4y = 0.$
3. $12; 9; 15; \frac{5}{4}; (0,\pm 15); (0,\pm 12); (\pm\frac{27}{4},\pm 15); \frac{27}{2}; 4x \pm 3y = 0.$
5. $4; 4; 4\sqrt{2}; \sqrt{2}; (0,\pm 4\sqrt{2}); (0,\pm 4); (\pm 4,\pm 4\sqrt{2}); 8; x \pm y = 0.$
7. $5; 2; \sqrt{29}; \frac{1}{5}\sqrt{29}; (\pm\sqrt{29},0); (\pm 5,0); (\pm\sqrt{29},\pm\frac{4}{5}); \frac{8}{5};$ $2x \pm 5y = 0.$
9. $8; 8; 8\sqrt{2}; \sqrt{2}; (\pm 8\sqrt{2},0); (\pm 8,0); (\pm 8\sqrt{2},\pm 8); 16; x \pm y = 0.$
17. $3y^2 - x^2 = 192.$
19. $9x^2 - 4y^2 = 144.$
21. $16x^2 - 9y^2 = 576.$
23. $x^2 - 3y^2 = 6.$
25. $9x^2 - 4y^2 = 156.$
27. $9x^2 - 3y^2 = 64.$
29. $3x^2 - y^2 = 44.$

Pages 107, 108

(Answers to Exercises 1–11 are given in the order of the questions.)

1. $(1,-2); (1 \pm \sqrt{5},-2); (3,-2), (-1,-2); 1\cdot x - 2y - 5 = 0,$ $x + 2y + 3 = 0; 4(y + 2)^2 - (x - 1)^2 = 4.$

3. $(-4,-3)$; $(-4,-3 \pm \sqrt{29})$; $(-4,-1)$, $(-4,-5)$; 25; $2x - 5y - 7 = 0$, $2x + 5y + 23 = 0$; $4(x+4)^2 - 25(y+3)^2 = 100$.

5. $(-1,-1)$; $(-1 \pm \sqrt{13},-1)$; $(-4,-1),(2,-1)$; $\frac{8}{3}$; $2x + 3y + 5 = 0$, $2x - 3y - 1 = 0$; $4x^2 - 9y^2 + 8x - 18y + 31 = 0$.

7. $(-2,3)$; $(-2 \pm \sqrt{10},3)$; $(-3,3)$, $(-1,3)$; 18; $3x + y + 3 = 0$, $3x - y + 9 = 0$; $y^2 - 9x^2 - 6y - 36x - 36 = 0$.

9. $(2,-3)$; $(2,-3 \pm \sqrt{5})$; $(2,-3 \pm \sqrt{3})$; $\frac{4}{3}\sqrt{3}$;
$\sqrt{3}x \pm \sqrt{2}y \pm 3\sqrt{2} - 2\sqrt{3} = 0$;
$3x^2 - 2y^2 - 12x - 12y - 12 = 0$.

11. $(-2,-3)$; $(-2 \pm 2\sqrt{2},-3)$; $(-4,-3)$, $(0,-3)$; 4; $x - y - 1 = 0$, $x + y + 5 = 0$; $x^2 - y^2 + 4x - 6y - 1 = 0$.

13. $8x^2 - y^2 - 112x + 384 = 0$.

15. $9x^2 - 4y^2 - 18x + 16y - 151 = 0$.

17. $5x^2 - 4y^2 + 10x - 8y - 79 = 0$.

19. $3x^2 - 4y^2 - 24x - 16y + 20 = 0$.

21. $96y^2 - 25x^2 - 768y + 100x - 964 = 0$.

23. $9x^2 - 16y^2 + 54x + 160y - 463 = 0$.

25. $3x^2 - y^2 - 6x + 4y - 13 = 0$. **27.** $x^2 - y^2 + 2x - 6y - 43 = 0$.

Pages 113, 114

1. $y + 3 = 0$; $12x - 5y - 39 = 0$. **3.** $x + 4y \pm 4\sqrt{10} = 0$.

5. $x + y \pm 2\sqrt{3} = 0$. **7.** $3x + 4y + 7 = 0$, $3x - 4y + 23 = 0$.

9. $x + y - 3 \pm 3\sqrt{2} = 0$. **11.** $4x - 3y + 6 \pm 3\sqrt{21} = 0$.

15. $m = m_1 + m_2$.

Pages 119, 120

3. $5\sqrt{2}$; $\frac{5}{2}\sqrt{2}$. **7.** $3x - 8y + 5 = 0$, $8x + 3y - 11 = 0$.

9. $\frac{4}{9}$; $\frac{9}{4}$. **11.** $x + y - 2 = 0$, $x - y - 2 = 0$.

Page 130

1. $13x^2 = 56250y$. **3.** 0.56 approx.; $\dfrac{x^2}{94556.25} + \dfrac{y^2}{65025} = 1$.

Pages 133, 134

7. $2x^2 - 2y^2 = a^2$.

15. $648x^2 + 729y^2 - 2160x - 1944y - 104 = 0$.

19. $\dfrac{a}{2}$.

Page 136

1. $2x'y' + 25 = 0.$
3. $2x' - 5\sqrt{2} = 0.$
5. $3x'^2 - y'^2 = 4.$
7. $x'^2 + 2x'y' + 2y'^2 = 1.$
9. $11x'^2 - 14y'^2 = 8.$
11. $(\frac{5}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}).$
13. $y'^2 = 8x'.$
15. $2y'^2 + 2\sqrt{3}x'y' - 1 = 0.$

Pages 143, 144

1. Ellipse; $3x''^2 + 2y''^2 = 6.$
3. Parabola; $y''^2 = 2\sqrt{2}x''.$
5. Hyperbola; $4x''^2 - 9y''^2 = 36.$
7. Lines; $5x''^2 - y''^2 = 0.$
9. Parabola; $y''^2 = 4x''.$
11. Hyperbola; $7y''^2 - 3x''^2 = 2.$
13. Hyperbola; $3y''^2 - 2x''^2 = 6.$
15. Ellipse; $2x''^2 + y''^2 = 4.$
17. Hyperbola; $12x'^2 - y'^2 = 5.$

Page 148

1. $x - y + 2 = 0.$
3. $xy = 6.$
5. $16x^2 + 9y^2 = 144.$
7. $x^2 = y^3.$
9. $y = 4x^2 + 32x + 64.$
11. $x^2 - y^2 = 4.$
13. $x - y + 1 = 0.$
15. $9y^2 = 8x^3.$
17. $x^2 + 2xy + y^2 + 2x - 2y = 0.$
19. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

Page 155

9. $P = 2\pi; A = 2.5.$
11. $P = 4; A = \frac{1}{2}$
13. $P = 4\pi; A = 5.$

Page 157

3. $(4, 120°), (4, 240°).$

Page 164

1. $x = 8.$
3. $x - \sqrt{3}y = 0.$
5. $x^2 + y^2 - 3x = 0.$
7. $x^4 + y^4 + 39y^2 + 2x^2y^2 + 16x^2y + 16y^3 - 25x^2 = 0.$
9. $24x^2 + 25y^2 - 20x - 100 = 0.$
11. $\sin\theta + \cos\theta = 0.$
13. $r = \pm 4.$
15. $r = 4(\cos\theta + \sin\theta).$
17. $r = 8 \csc\theta \cot\theta.$
19. $r^2 = 24 \csc 2\theta.$

Page 170

3. $z; y; x$
5. $\sqrt{29}.$
7. z-axis; y-axis; x-axis.
9. $\sqrt{57}.$
13. $14x - 12y + 16z + 15 = 0.$

Pages 174, 175

1. $\dfrac{1}{\sqrt{35}}; \dfrac{5}{\sqrt{35}}; \dfrac{3}{\sqrt{35}}.$

3. $\dfrac{4}{\sqrt{77}}; \dfrac{-6}{\sqrt{77}}; \dfrac{5}{\sqrt{77}}.$

5. $\dfrac{-3}{\sqrt{155}}; \dfrac{-5}{\sqrt{155}}; \dfrac{11}{\sqrt{155}}.$

7. $\dfrac{2}{3\sqrt{5}}; \dfrac{-5}{3\sqrt{5}}; \dfrac{4}{3\sqrt{5}}.$

9. $\dfrac{-2}{\sqrt{65}}; \dfrac{-5}{\sqrt{65}}; \dfrac{6}{\sqrt{65}}.$

11. $\dfrac{15}{\sqrt{949}}; \dfrac{20}{\sqrt{949}}; \dfrac{-18}{\sqrt{949}}.$

15. $\dfrac{1}{\sqrt{3}}; \dfrac{1}{\sqrt{3}}; \dfrac{1}{\sqrt{3}}.$

17. 1,0,0; 0,1,0; 0,0,1.

Pages 178, 179

1. $x + y + \sqrt{2}z - 10 = 0.$

3. $2x + y - 2z - 9 = 0.$

5. $-\dfrac{5}{\sqrt{38}}x - \dfrac{2}{\sqrt{38}}y + \dfrac{3}{\sqrt{38}}z - \dfrac{4}{\sqrt{38}} = 0$

9. $2x - 5y - 10 = 0;\ x + 2z - 5 = 0;\ 5y - 4z + 10 = 0.$

11. $Ax + D = 0;\ By + D = 0;\ Cz + D = 0.$

17. $2x - 3y + 9z - 34 = 0.$

19. $\left(\dfrac{-23}{34}, \dfrac{-115}{68}, \dfrac{111}{68}\right).$

21. $4x + 10y + 6z - 13 = 0.$

Pages 183, 184

1. $\dfrac{x-3}{2} = \dfrac{y-2}{2} = \dfrac{z+4}{-2}$ or $\dfrac{x-5}{-2} = \dfrac{y-4}{-2} = \dfrac{z+6}{2}$

3. $\dfrac{x-4}{3} = \dfrac{y-5}{4} = \dfrac{z-6}{5}.$

5. $\dfrac{x-2}{2} = \dfrac{y-4}{-3} = \dfrac{z-3}{3}.$

7. $\dfrac{x-3}{-1} = \dfrac{y-4}{\sqrt{2}} = \dfrac{z+5}{1}.$

9. $\dfrac{x-1}{6} = \dfrac{y+2}{7} = \dfrac{z-3}{-5}.$

11. $\dfrac{4}{3\sqrt{5}}; \dfrac{-5}{3\sqrt{5}}; \dfrac{2}{3\sqrt{5}}.$

13. $\dfrac{3}{\sqrt{83}}; \dfrac{7}{\sqrt{83}}; \dfrac{5}{\sqrt{83}}.$

15. XY: (1,0,0); YZ: $(0,\frac{7}{2},2)$; XZ: (1,0,0).

Pages 187, 188

1. $x^2 + y^2 + z^2 - 4x - 6y - 8z + 4 = 0.$
3. $4x^2 + 4y^2 + 4z^2 + 12x - 4y - 8z - 11 = 0.$
5. $(-3,2,-7)$; 4. **7.** $(2,-1,4)$; $\frac{5}{2}$.
9. $x^2 + y^2 + z^2 - 6x + 8y - z + 17 = 0.$

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